Calculating QCD I:

Can we simply apply the appropriate Feynman rules and correct the parton model?

Yes, but is ain't easy! QCD Feynman diagrams exhibit infinities at nearly every turn.***

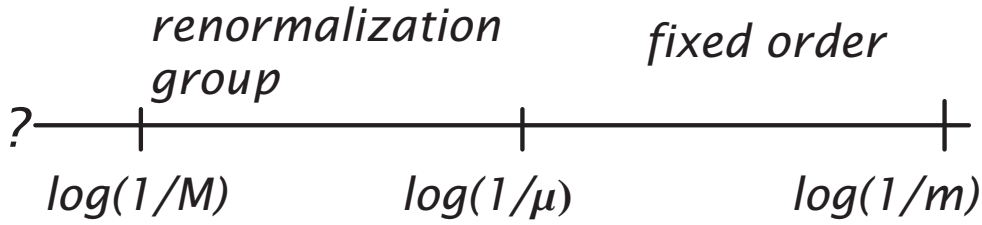
First consider life in the Ultra-Violet – short distance/times or large momenta (the Renormalization Group at work):

The UV singularities mean that the theory

- does *not* specify the strength of the coupling in terms of the "bare" coupling in the Lagrangian
- does specify how the coupling varies with scale [$\alpha_s(\mu)$ measures the "charge inside" a sphere of radius $1/\mu$]

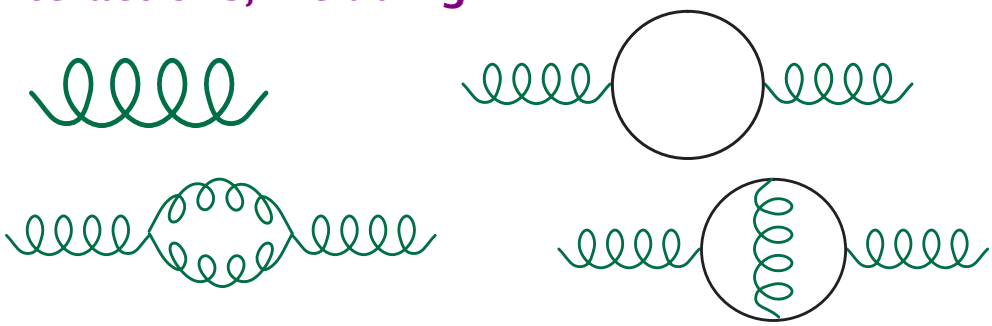
Consider a range of distance/time scales –

- use the renormalization group below some scale μ (perhaps down to a GUT scale M)
- use fixed order perturbation theory above (until reaching the hadronic scale m where things become non-perturbative)



*** Typical of any renormalizable gauge field theory. This is one reason why String theorists want to study something else! We will not discuss the issue of choice of gauge. Typically axial gauges (n•A=0) yield diagrams that are more parton-model-like, so-called physical gauges.

Corrections to the parton model come from adding gluon interactions, including



Loops are UV divergent like dk^4/k^4 - keep contributions from the range μ to M.

The (cutoff) calculation yields results that can be organized as an effective (renormalized) coupling**

$$\alpha_s(\mu) \sim \alpha_s(M) - \frac{\beta_0}{4\pi}\ln\left[\frac{\mu^2}{M^2}\right]\alpha_s{}^2(M) + \left(\frac{\beta_0}{4\pi}\right)^2 \ln^2\left[\frac{\mu^2}{M^2}\right]\alpha_s{}^3(M) + \ldots$$

$$\alpha_s(\mu) = \frac{\alpha_s(M)}{1 + \left(\beta_0/4\pi\right)\alpha_s(M)\ln\left[\frac{\mu^2}{M^2}\right]}\,;\ \beta_0 = \frac{11C_A - 2n_f}{3} = 11 - \frac{2}{3}n_f$$

This result is more compactly specified by the renormalization group equation

$$\mu^2\frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s) = -\alpha_s\sum_{n=0}^{\infty}\beta_n\left(\frac{\alpha_s}{4\pi}\right)^{n+1};\ \beta_1 = 102 - \frac{38}{3}n_f, \ldots$$

The first form above is the "one-loop" solution for α_s (keeping only the β_0 term). pQCD allows one to systematically include the higher loop corrections.

**Masses and wave functions also exhibit renormalization.

Dimensional Regularization – a brief interlude

Want -

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left(k^2 - m^2\right)^2}$$

Consider – [4→4-2ε, Wick rotate to Euclidean space]

$$\int \frac{d^4k}{(2\pi)^4} \to (\mu)^{2\varepsilon} \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \to (\mu)^{2\varepsilon} \int \frac{d\Omega_{4-2\varepsilon}}{(2\pi)^{4-2\varepsilon}} \int dk_E k_E^{3-2\varepsilon}$$

Calculate –

$$\int \frac{d\Omega_{4-2\varepsilon}}{(2\pi)^{4-2\varepsilon}} = \frac{2}{(4\pi)^{2-\varepsilon}} \frac{1}{\Gamma(2-\varepsilon)}$$

$$(\mu)^{2\varepsilon} \int_0^\infty dk_E \frac{k_E^{3-2\varepsilon}}{\left(k_E^2 + m^2\right)^2} = \frac{(\mu)^{2\varepsilon}}{2(m)^{2\varepsilon}} \int_0^1 dz z^{1-\varepsilon} (1-z)^{\varepsilon-1}$$

$$= \frac{1}{2}\left(\frac{\mu}{m}\right)^{2\varepsilon} \frac{\Gamma(\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(2)}$$

or, using $\Gamma(1+z) = z\Gamma(z); \Gamma'(1) = -\gamma_E = -0.5772\ldots$

$$I = \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}\left(\frac{\mu}{m}\right)^{2\varepsilon} \underset{\varepsilon\to 0}{\to} \frac{1}{(4\pi)^2}\left[2\ln\left(\mu/m\right) + \frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + O(\varepsilon)\right]$$

$\overline{MS}$ Scheme – subtract $1/\varepsilon$, γ_E and $\ln(4\pi)$ bits****

**** You can hid anything in infinity!

The "running" coupling illustrates typical features of QCD –

- expanding to a fixed power of α_s is often not enough
- large logarithms (the remnants of the infinities) must be resumed to *all* orders by some technique

By measuring α_s at some scale μ_0 a parameter Λ_{QCD} can be defined.

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln\left(\mu^2 / \Lambda_{QCD}^2\right)}; \quad \Lambda_{QCD} = \mu_0 e^{-\frac{2\pi}{\beta_0 \alpha_s(\mu_0)}}$$

Our knowledge of the behavior in the UV is now encoded in Λ_{QCD}. (Note that the value of Λ_{QCD} will to depend on the order of the β function used (1-loop, 2-loop, etc.). The data does not change, only the internal theoretical parameters.)

Experimentally $\Lambda_{QCD} \sim 200$ MeV

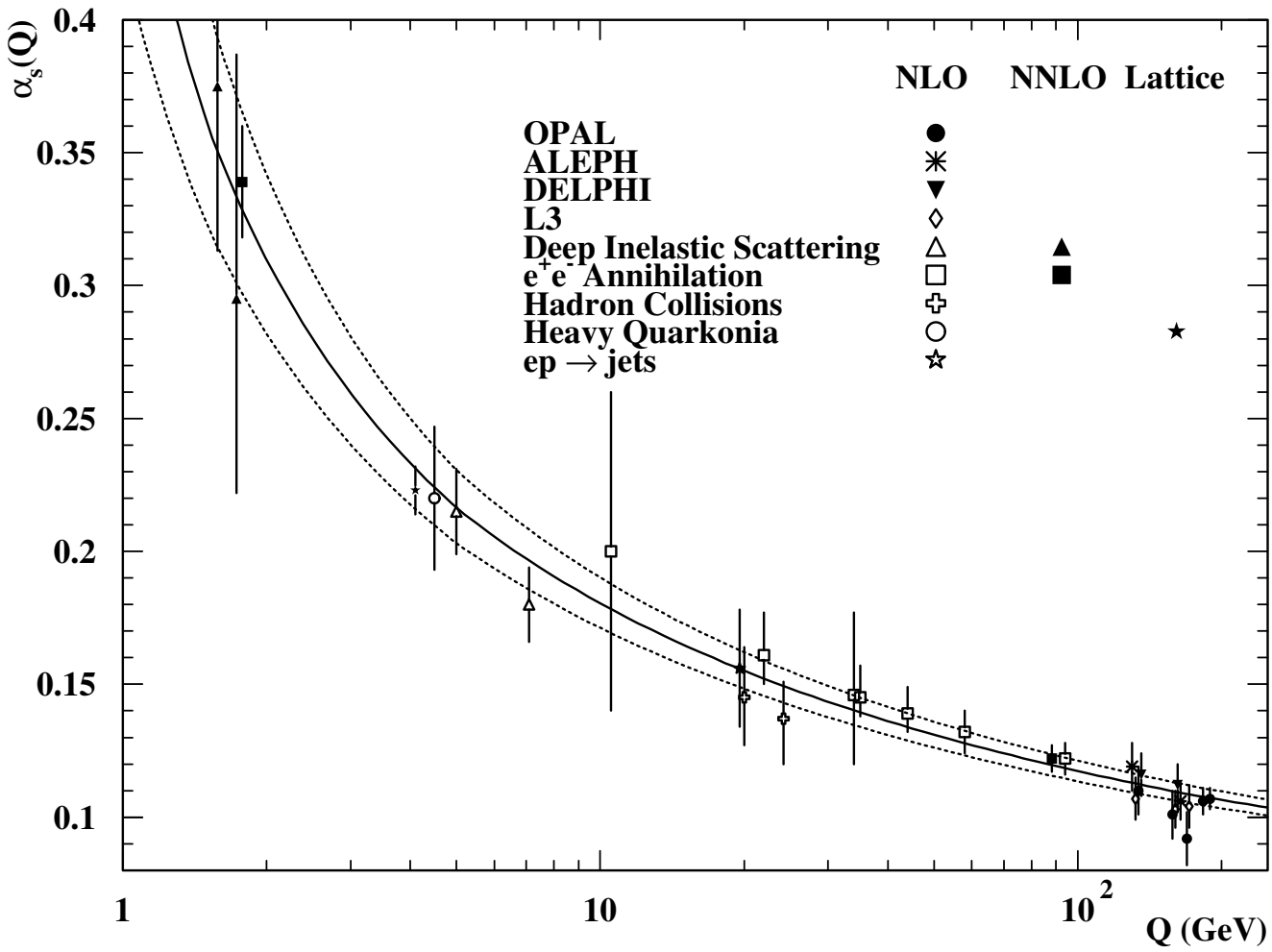
The running of the coupling is clear in the data, as is the precision of our knowledge of α_s, e.g., the winter 2000 number is $\alpha_s(m_Z) = 0.119 \pm 0.004$.

Look at the (amazing!) behavior of running α_s –

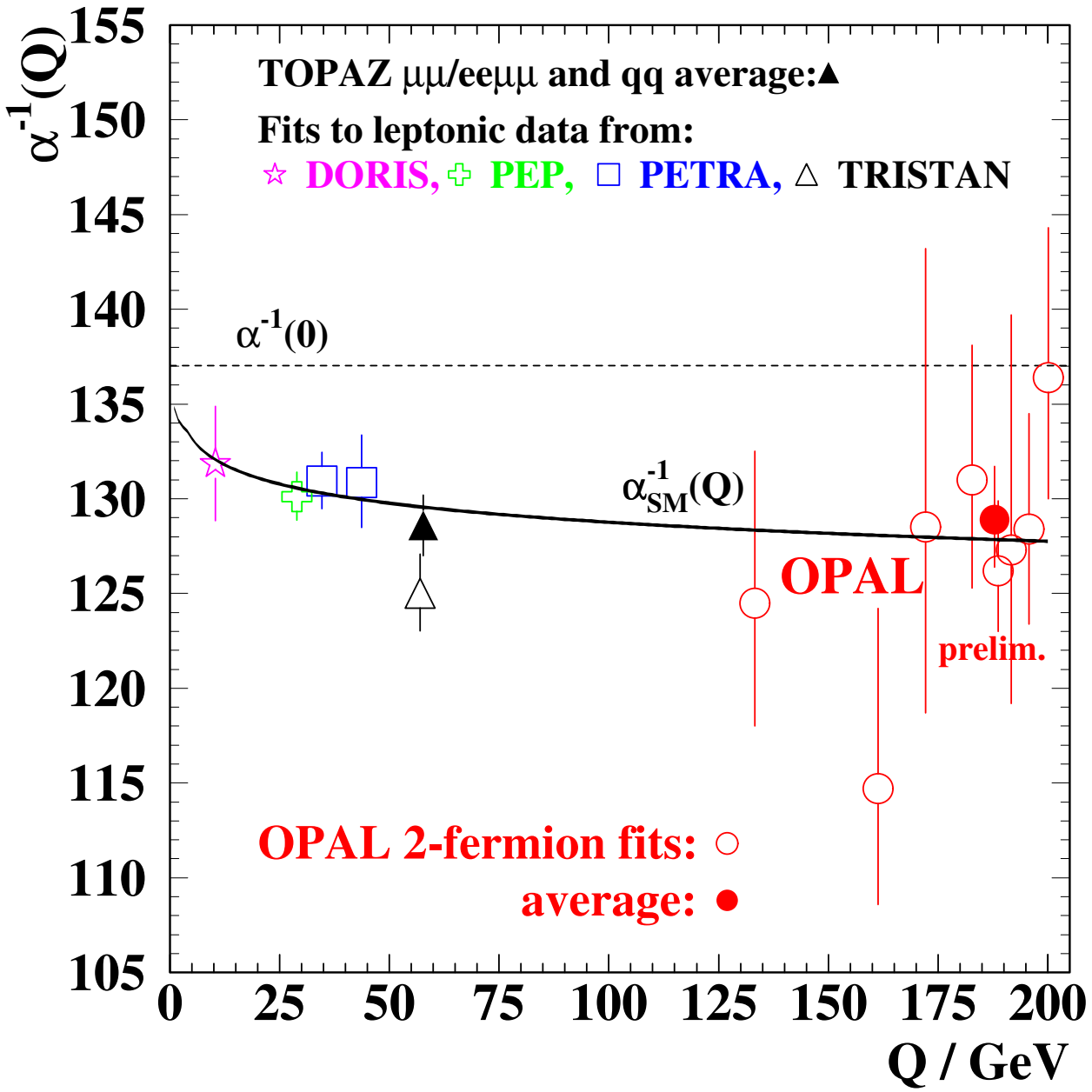
- As μ increases, α_s decreases – asymptotic freedom!
- As μ decreases, α_s increases – infrared slavery!

Just what we wanted in the parton model!!!!!

$\alpha_s(Q)$


$\alpha_s(Q)$
0.4
0.35
0.3
0.25
0.2
0.15
0.1
1
10
10^2
Q (GeV)
NLO
NNLO
Lattice
OPAL
ALEPH
DELPHI
L3
Deep Inelastic Scattering
e^+e^- Annihilation
Hadron Collisions
Heavy Quarkonia
ep → jets

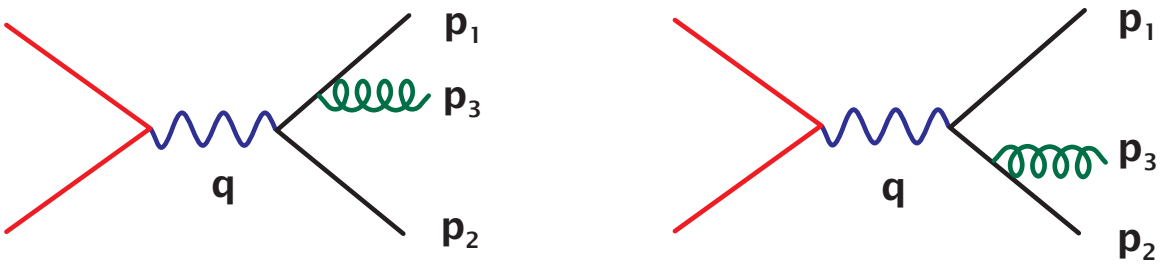
$\alpha_{EM}(Q)$



Runs the other way!! ($\beta_0^{EM} < 0$!)

pQCD Calculation II:

Revisit e^+e^- scattering (massless partons!)



Define handy variables ($q^2=Q^2=s$)

$$x_i = \frac{2E_i}{\sqrt{q^2}} = \frac{2p_i \cdot q}{q^2} \Rightarrow x_i \geq 0 \Rightarrow \sum_{i=1}^{3} x_i = 2$$

$$1-\cos\theta_{ij} = \frac{p_i \cdot p_j}{E_i E_j} = \frac{(q-p_k)^2}{2E_i E_j} = \frac{2(1-x_k)}{x_i x_j}$$

The cross section looks like****

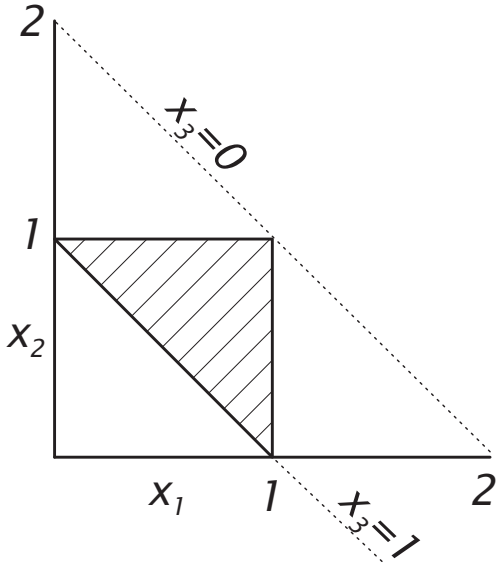
$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \Leftarrow \sigma_0 = \frac{4\pi\alpha^2}{3q^2} \cdot 3\sum_f e_f^2$$

Hence the singular regions are:

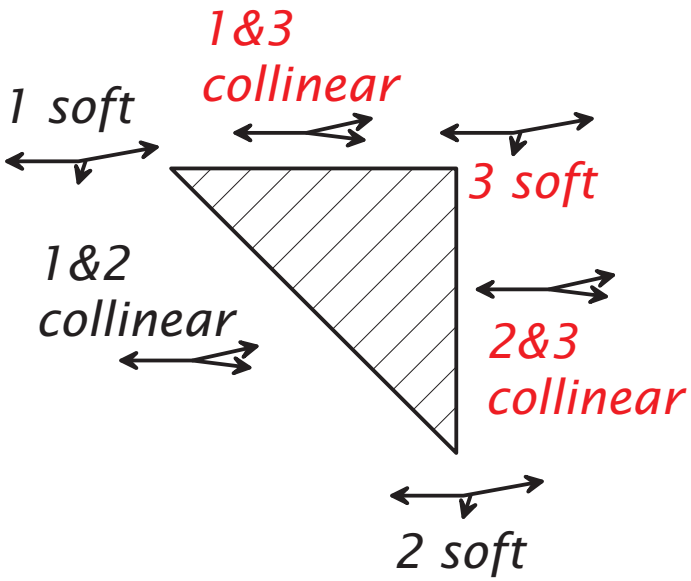
- Collinear gluon -- $\theta_{13}\to 0$, $x_2\to 1$
  $\theta_{23}\to 0$, $x_1\to 1$
- Soft gluon – $x_3\to 0 \to (x_1\to 1$ and $x_2\to 1)$ with $(1- x_1)/(1- x_2)$ fixed

**** Calculate this at home!

Thus the phase space looks like:



with boundaries being interesting



The red singularities arise from a propagator above going on-shell – either 1+3 or 2+3 !

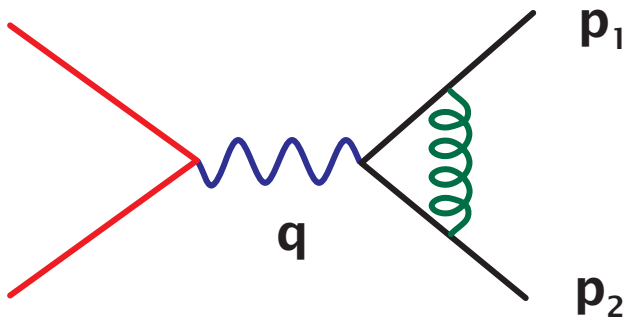
On-shell propagators – long distance propagation – perturbative expansion fails – 1+ α_s x big + α_s^2 x bigger …

Still parton model-like picture – short distance/time simple, long distance complex.

How do we proceed?

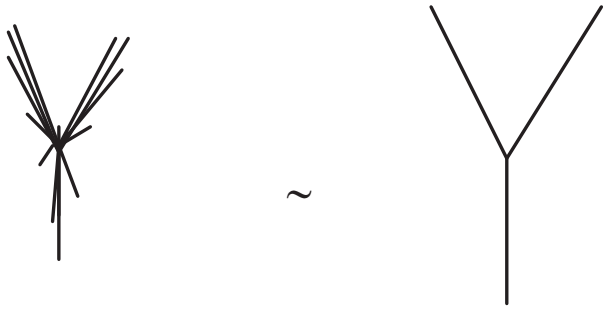
- **Ask questions that are insensitive to long distance structure**

  e.g., σ_{TOT} which receives contributions from all states – details cannot matter – in detail the singularities in the virtual graph cancel with those above



- **More generally define InfraRed Safe (IRS) quantities – insensitive to collinear and soft emissions, i.e.,**

$$S_{n+1}(p_1,\ldots,(1-\lambda)p_n,\lambda p_{n+1}) = S_n(p_1,\ldots,p_n)$$

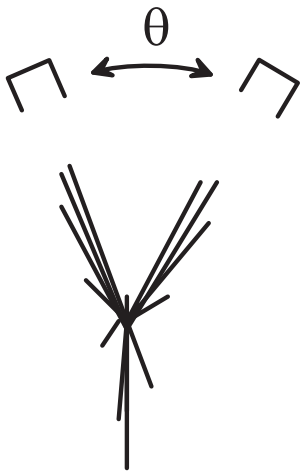


Thrust - dσ/dT

$$S_n(p_1,\ldots,p_n) = \delta(T - T_n(p_1,\ldots,p_n))$$

$$T_n(p_1,\ldots,p_n) = \max_{\vec{u}} \frac{\sum_{i=1}^{n} |\vec{p}_i \cdot \vec{u}|}{\sum_{i=1}^{n} |\vec{p}_i|}$$

EEC Energy-Energy Correlation



dΣ/dcosθ

$$S_n(p_1,\ldots,p_n)=\Sigma=\sum_{ij}\frac{E_iE_j}{q^2}\delta(\cos\theta_{ij}-\cos\theta)$$

Both quantities are insensitive to:

- $E_i, p_i \to 0$
- Collinear split $E_n, p_n \to (1-\lambda)(E_n, p_n) + \lambda(E_n, p_n)$ (even for the autocorrelation $E_n^2 = E_n^2(1-\lambda)^2 + E_n^2\lambda^2 + 2\lambda(1-\lambda)E_n^2$)

Jet cross sections also qualify and we will come back to them.

Check limits:

- Collinear & soft limit [~2 jets] - potentially 2 logs!

$$\frac{1}{\sigma}\frac{d\sigma}{dT}\xrightarrow[T\to 1]{}\frac{\alpha_s}{2\pi}C_F\frac{\ln(1/(1-T))}{1-T}$$

Back-to-Back EEC

$$\frac{1}{\sigma}\frac{d\Sigma}{d\cos\theta}\xrightarrow[\theta\to\pi]{}\frac{\alpha_s}{2\pi}C_F\frac{\ln(2/(1+\cos\theta))}{1+\cos\theta}$$

- **Collinear only - Front-to-Front – potential single log**

$$\frac{1}{\sigma}\frac{d\Sigma}{d\cos\theta} \xrightarrow{\theta\to 0} \frac{3\alpha_s}{8\pi} C_F \frac{1}{1-\cos\theta}$$

In the Double Leading Log Approximation – sum the large double logarithms [soft & collinear] order by order (often in impact parameter space) – corresponding to ordered emission of gluons -

Let κ be a dimensionless parameter controlling this limit and L=ln(κ) – a typical perturbative series or factor

$$P(\kappa) \sim 1 - \frac{\alpha_s}{2\pi} C_X L^2 + \frac{1}{2}\left(\frac{\alpha_s}{2\pi} C_X\right)^2 L^4 - \frac{1}{6}\left(\frac{\alpha_s}{2\pi} C_X\right)^3 L^6 + \ldots$$

$$\to \exp\left[-\frac{\alpha_s}{2\pi} C_X L^2\right]$$

This structure arises often

- **EEC –** $L = \ln\left(\frac{1+\cos\theta}{2}\right); C_X = C_F$**, T -** $L = \ln(1-T); C_X = 2C_F$

and is associated with

- **the probability to *not* radiate soft and collinear gluons, i.e., very small if soft and collinear enough**
- **the name Sudakov Form Factor**

In the limit $\kappa \to 0$, subleading terms will be important!

PQCD Calculation III:

Apply Dim-Reg to total cross section: $\sigma_0 = \dfrac{4\pi\alpha^2}{3Q^2}\cdot 3\sum_f e_f^2$

- **Real**

$$\sigma^{q\bar{q}g}(\varepsilon) = \sigma_0 H(\varepsilon)\int dx_1 dx_2 \frac{C_F\alpha_s(\mu)}{2\pi}\times$$

$$\left[\frac{(1-\varepsilon)(x_1^2+x_2^2)-\varepsilon(1-x_1-x_2)}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1+\varepsilon}(2-x_1-x_2)^{\varepsilon}} - 2\varepsilon\right]$$

$$= \sigma_0 H(\varepsilon)\frac{C_F\alpha_s(\mu)}{2\pi}\left[\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}-\pi^2+O(\varepsilon)\right]$$

$$H(\varepsilon) = \frac{3(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} = 1+O(\varepsilon)$$

- **Virtual**

$$\sigma^{q\bar{q}(g)}(\varepsilon) = \sigma_0 H(\varepsilon)\frac{C_F\alpha_s(\mu)}{2\pi}\left[-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}-8+\pi^2+O(\varepsilon)\right]$$

- **Sum and set $\varepsilon \to 0$, R = $\sigma(e^+e^-\to\text{hadrons})/\sigma(e^+e^-\to\mu^+\mu^-)$**

$$R = 3\sum_f e_f^2\left[1+\frac{3C_F\alpha_s(\mu)}{4\pi}+O(\alpha_s^2)\right] = 3\sum_f e_f^2\left[1+\frac{\alpha_s(\mu)}{\pi}+O(\alpha_s^2)\right]$$

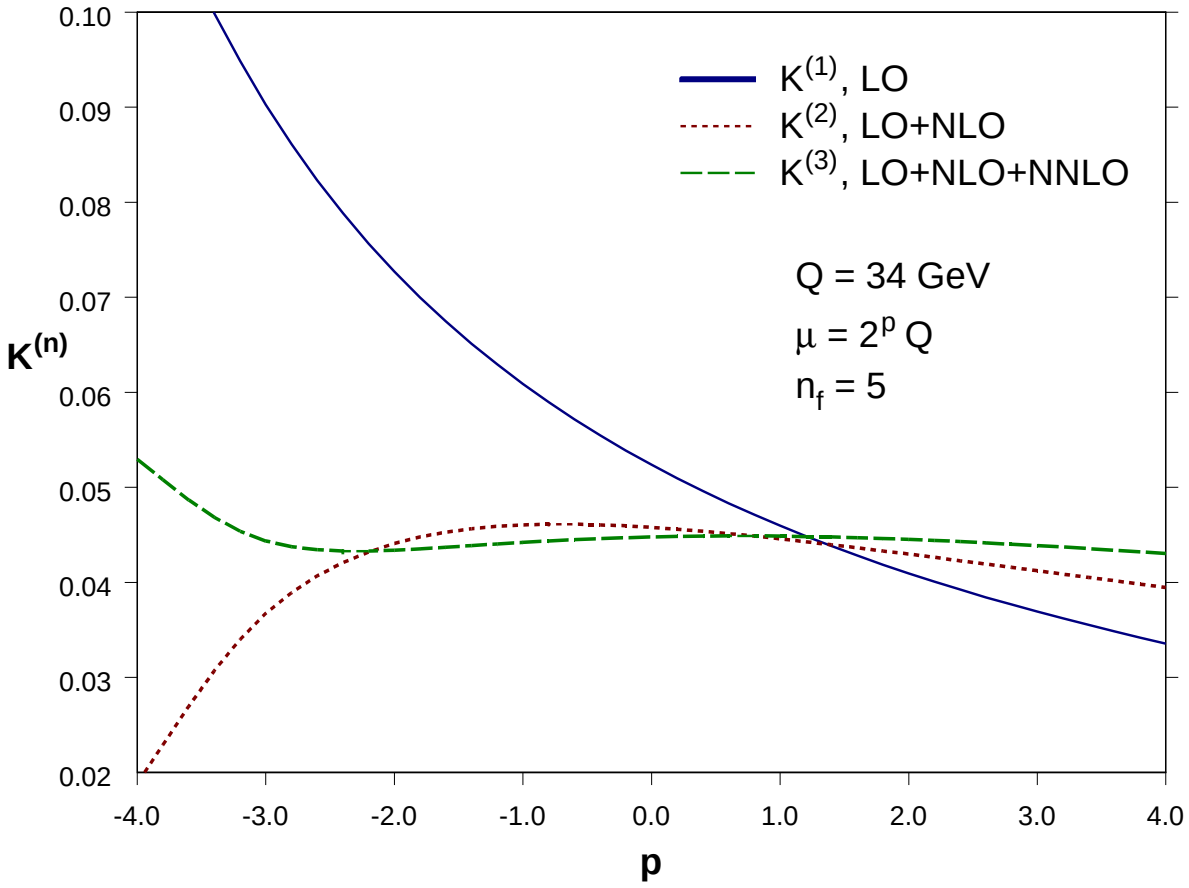
More work –

$$K = \frac{R}{3\sum_f e_f^2} - 1 = \frac{\alpha_s(\mu)}{\pi} + \left\{1.4092 + 1.9167 \ln\left(\frac{\mu^2}{Q^2}\right)\right\}\left(\frac{\alpha_s(\mu)}{\pi}\right)^2$$
$$+ \left\{-12.805 + 7.8179 \ln\left(\frac{\mu^2}{Q^2}\right) + 3.674 \ln^2\left(\frac{\mu^2}{Q^2}\right)\right\}\left(\frac{\alpha_s(\mu)}{\pi}\right)^3$$
$$+ O\left(\left(\frac{\alpha_s(\mu)}{\pi}\right)^4\right)$$

Typical Behavior –

- Physical quantity is μ independent
- higher orders exhibit explicit $\ln(\mu^2/Q^2)$ terms
- Σ of higher orders exhibits reduced dependence on unphysical parameter μ
- at order α_s^n the residual $\ln(\mu^2)$ dependence is order α_s^{n+1}
- μ dependence is an artifact of the truncation of the perturbative expansion

Plot K



Lessons –

- **Big improvement LO → LO + NLO**
- **See stable region around p=0, μ~Q ("natural" scale)**
- **Sign of μ→0 divergence oscillates order by order**

Prescription for μ?

- **PMS (Principle of Minimal Sensitivity)** $\mu \left. \frac{dK}{d\mu} \right|_{\mu_{PMS}} = 0$
- **FAC (Fastest Apparent Convergence)** $K^{(1)}\left(\mu_{FAC}\right) = K^{(2)}\left(\mu_{FAC}\right)$
- **BLM – absorb all n_f dependence into α_s**

But there is no *right* answer!

Residual dependence on μ in the "physical range" (e.g., p=0±1) is simply a (crude) measure of the uncertainty due to the uncalculated higher order contributions!