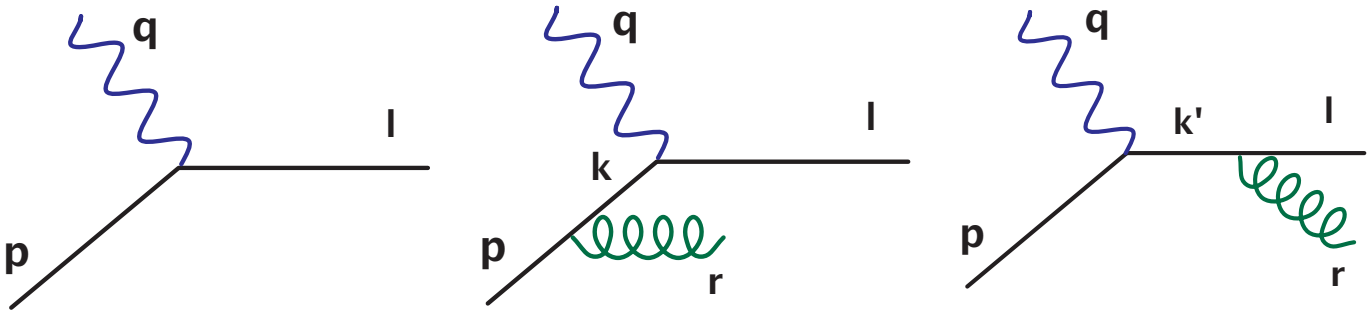
pQCD Calculation III - Parton Distribution Functions in QCD

(see G. Sterman and J. Stirling)

Revisit DIS – include real gluon emission (massless partons)



Singularities arise when the internal propagators go on-shell (collinear and soft gluon emission).

In the appropriate (light-cone) gauge, the divergent contribution in the middle diagram can be written*

$$\hat{F}_2\Big|_{Div} = e_q^2 \frac{\alpha_s}{2\pi} x\hat{P}_{qq}(x)\int_0^{2\nu} \frac{d|k^2|}{|k^2|}$$

- the $|k^2|$ integral goes all the way up to the kinematic boundary – it is not cutoff at a fixed (small) value as assumed by the parton model (so expect some differences)
- the $|k^2|$ integral is singular at the lower limit – control with a cutoff κ^2 for now (this "long distance" behavior is controlled by "confinement" in real life)

* This is gauge dependent - only the sum of the middle and left graph squared is gauge invariant. In the light-cone gauge only the middle graph is singular.

- the (collinear) singularity is multiplied by a characteristic function of the quark's momentum fraction *x* – the "splitting function" – that tells us how the longitudinal momentum is shared

$$\hat{P}_{qq}(x) = C_F \frac{1+x^2}{1-x}$$

Put in cutoff and include *all* diagrams above (in standard from)

$$\hat{F}_2(x,Q^2) = e_q^2 x\left[\delta(1-x) + \frac{\alpha_s}{2\pi}\left(\hat{P}_{qq}(x)\ln\left(\frac{Q^2}{\kappa^2}\right) + C(x)\right)\right]$$

where both P(x) and the finite bit C(x) are calculable functions in pQCD.

Conclude!

- Scaling is broken by $\ln(Q^2)$ terms (and we must sum them)!

The distribution of quarks (in a quark) is now

$$q_q(x,Q^2) = \delta(1-x) + \frac{\alpha_s}{2\pi}\left(\hat{P}_{qq}(x)\ln\left(\frac{Q^2}{\kappa^2}\right) + C(x)\right)$$

Some calculational details: chose the following vectors for the incident quark, light-like gauge fixing vector and virtual photon:

see Chapter 4 in

$$p^\mu = (P,0,0,P);\ n^\mu = \left(\frac{1}{2P},0,0,\frac{-1}{2P}\right);\ q^\mu = \nu n^\mu + q_T^\mu$$

such that

$$p^2 = n^2 = q_T \cdot n = q_T \cdot p = 0;\ n \cdot p = 1;\ q \cdot p = \nu$$

$$q_T^2 = -q^2 = Q^2;\ x = Q^2/2\nu$$

If the emitted gluon has momentum r^μ and polarization ε^μ, we require (conserved current and gauge choice)

$$\varepsilon \cdot r = \varepsilon \cdot n = 0$$

The momentum of the internal quark leg can be written in terms of a transverse vector k_T (similar to q_T)

$$k^\mu = \xi p^\mu + \frac{k_T^2 - |k^2|}{2\xi} n^\mu + k_T^\mu;\ d^4k = \frac{d\xi}{2\xi} dk^2 d^2k_T$$

The appropriately summed, averaged (spin and color) and projected matrix element is

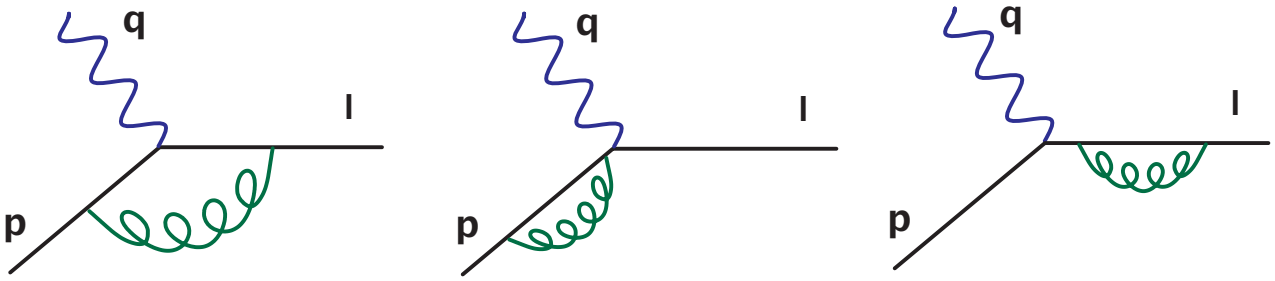
$$\frac{1}{4\pi} n^\alpha n^\beta \overline{\Sigma} |M|^2_{\alpha\beta} = \frac{8e_q^2\alpha_s}{|k^2|}\xi P(\xi)$$ with P(ξ=x) above!

The 2-body phase space in these variables is

$$d\Phi_2 = \frac{1}{16\nu\pi^2}\int d\xi\, dk^2 dk_T^2 d\theta\, \delta\left(k_T^2 - (1-\xi)|k^2|\right)\delta\left(\xi - x - \frac{|k^2| + 2q_T \cdot k_T}{2\nu}\right)$$

Performing all the integrals ($0<\theta<\pi$) except $d\xi$ yields the result above (the δ fcts put the outgoing q and g on-shell).

Next include virtual graphs, contribute in the soft limit



- ~$\delta((p+q)^2)$ - Contribute for x=1, $\rightarrow \delta(1-x)$ term + …
- Quark (baryon) number is conserved*, independent of Q^2

$$\hat{P}_{qq}(x) \rightarrow P_{qq}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]$$

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} \equiv \int_0^1 dx \frac{f(x)-f(1)}{(1-x)}$$

With care taken below for the process $g \rightarrow q\bar{q}$, this is just

$$\hat{P}(x) \rightarrow \hat{P}(x)_+ = P(x)$$

For a quark in a proton we introduce a "bare" quark distribution q_0 and convolute with above

$$q(x,Q^2) = q_0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi} q_0(\xi)\left[P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\kappa^2}\right) + C_q\left(\frac{x}{\xi}\right)\right] + \ldots$$

q_0 plays similar role to $\alpha_s(M^2)$ used earlier - a place to hide infinities.

*Confirm quark number conservation with this form.

Introduce a *factorization* scale μ – "absorb" collinear singularities (for $|k^2|<\mu^2$) into the bare distribution and obtain the renormalized distribution, *i.e.*, the long distance physics is now all in the renormalized distribution

$$q(x,Q^2)=\int_x^1 \frac{d\xi}{\xi} q\left(\frac{x}{\xi},\mu^2\right)\left[\delta\left(1-\frac{x}{\xi}\right)+\frac{\alpha_s}{2\pi}\left\{P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\mu^2}\right)+\overline{C}_q\left(\frac{x}{\xi}\right)\right\}+\ldots\right]$$

The $\overline{C}_q$ term, and thus the parton distribution function, will depend on the factorization scheme, *i.e.*, how much of the original finite piece gets absorbed with the infinity.

Physical quantities are scheme independent and the calculation will be also if <u>all</u> parts are performed in the same scheme!

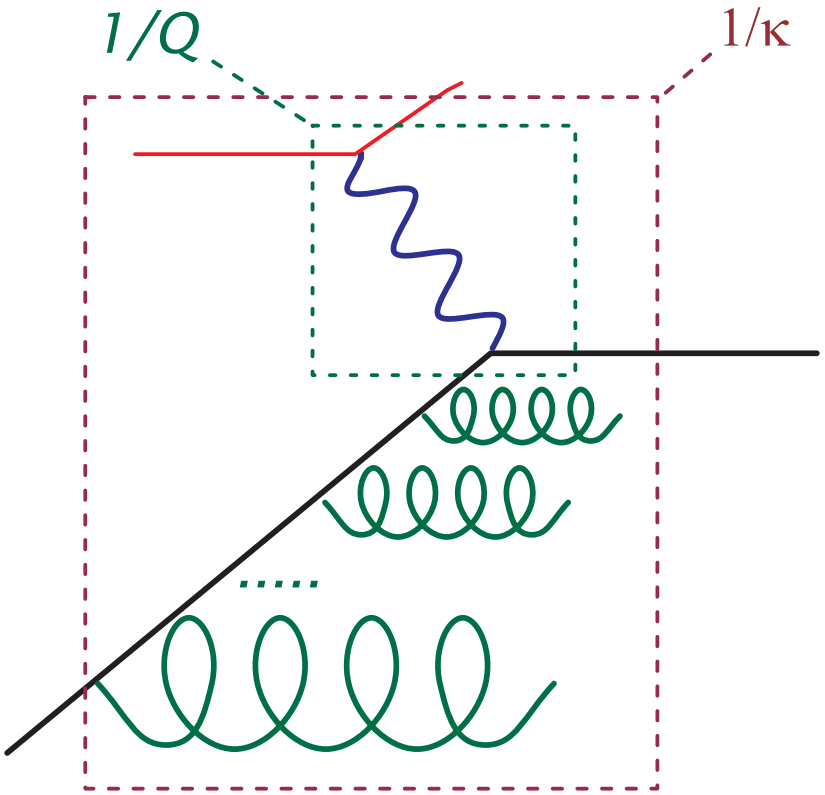
- The DIS choice is to absorb everything, $\overline{C}_q^{DIS}=0$

To review – we are summing the largest contributions of the emission of multiple gluons

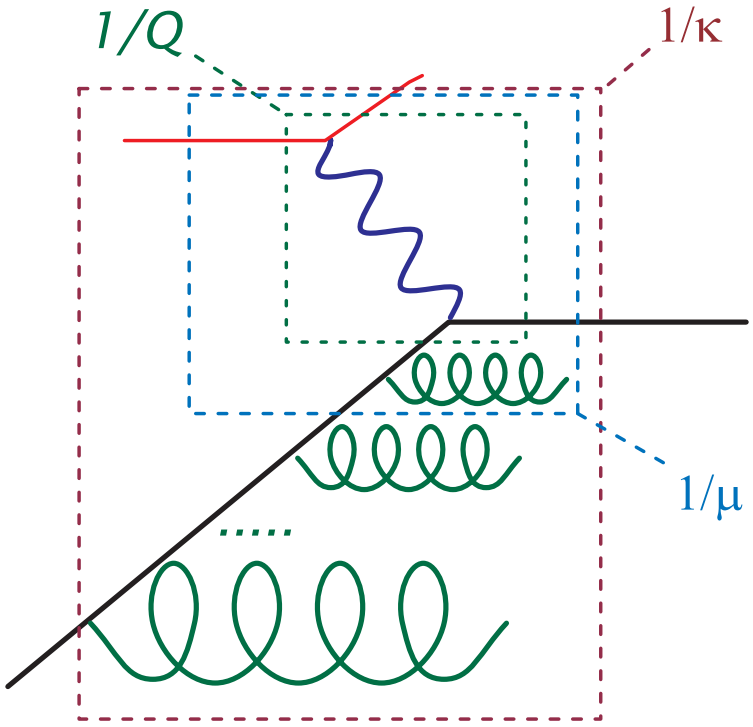
The change in size of the gluons represents the strong ordering of the transverse momenta

$$k_{T,1}^2 << k_{T,2}^2 << \cdots << k_{T,n}^2$$

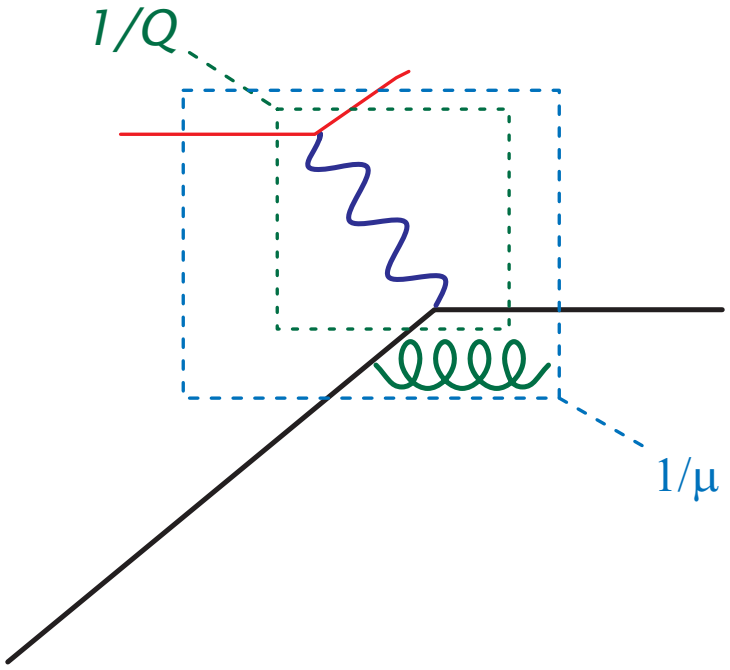
(smaller wavelength means larger momentum)



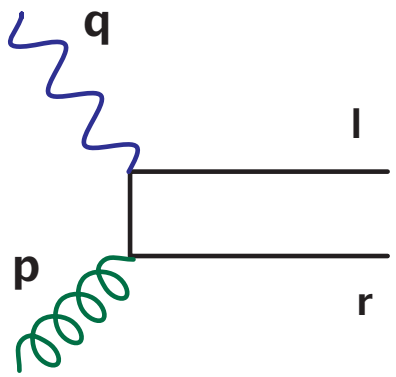
Now separate contributions above and below the factorization scale μ



And factor scales κ to μ into the renormalized distribution – leaving



At this order we also have (a quark from a gluon)



Or

$$\hat{F}_2^g\left(x,Q^2\right)=x\sum_{q,\bar{q}}e_q^2\,\frac{\alpha_s}{2\pi}\left[P_{qg}\left(x\right)\ln\left(\frac{Q^2}{\kappa^2}\right)+C_g\left(x\right)\right]$$

$$P_{qg}\left(x\right)=T_R\left[x^2+\left(1-x\right)^2\right];\quad T_R=\frac{1}{2}$$

So we really want a "bare" gluon distribution too -

$$q\left(x,Q^2\right)=q_0\left(x\right)+\frac{\alpha_s}{2\pi}\int_x^1\frac{d\xi}{\xi}\,q_0\left(\xi\right)\left[P_{qq}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\kappa^2}\right)+C_q\left(\frac{x}{\xi}\right)\right]+$$

$$+\frac{\alpha_s}{2\pi}\int_x^1\frac{d\xi}{\xi}\,g_0\left(\xi\right)\left[P_{qg}\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\kappa^2}\right)+C_g\left(\frac{x}{\xi}\right)\right]+\ldots$$

Hence factorizing with the choice $Q^2=\mu^2$ all is "hidden"

$$q(x,\mu^2)=\int_x^1 \frac{d\xi}{\xi} q(\xi,\mu^2)\left[\delta\left(1-\frac{x}{\xi}\right)+\frac{\alpha_s(\mu^2)}{2\pi}C_q^{\overline{MS}}\left(\frac{x}{\xi}\right)+\ldots\right]+$$
$$+\int_x^1 \frac{d\xi}{\xi} g(\xi,\mu^2)\left[\frac{\alpha_s(\mu^2)}{2\pi}C_q^{\overline{MS}}\left(\frac{x}{\xi}\right)+\ldots\right]$$

Again, the specific form of the "finite" piece, C(x) [called the *coefficient function*], depends on the renormalization scheme and on the specific quantity being calculated [*i.e.*, different for F_1 and F_2].

For the proton in $\overline{MS}$ ($\mu^2=Q^2$)

$$F_2(x,Q^2)=x\sum_{q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} q(\xi,Q^2)\left[\delta\left(1-\frac{x}{\xi}\right)+\frac{\alpha_s(Q^2)}{2\pi}C_q^{\overline{MS}}\left(\frac{x}{\xi}\right)+\ldots\right]+$$
$$+x\sum_{q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} g(\xi,Q^2)\left[\frac{\alpha_s(Q^2)}{2\pi}C_q^{\overline{MS}}\left(\frac{x}{\xi}\right)+\ldots\right]$$

As with the renormalized, running coupling, pQCD

- **does *not* tell us about the full parton distributions**
- **does tell us how they *evolve* with the scale μ**

We can express this information in an integral-differential evolution equation

The DGLAP Equation – (Dokshitzer-Gribov-Lipatov-Altarelli-Parisi)

$$\mu^2 \frac{\partial}{\partial \mu^2} q(x,\mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{d\xi}{\xi} P\left(\frac{x}{\xi}\right) q(\xi,\mu^2)$$

$$= \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{dz}{z} P(z) q\left(\frac{x}{z},\mu^2\right)$$

The splitting function P (like the β function) is what is calculable in pQCD.

$$P\left(z,\alpha_s(\mu^2)\right) = P^{(0)}(z) + \frac{\alpha_s(\mu^2)}{2\pi} P^{(1)}(z) + \ldots$$

The splitting function $P_{ab}(z)$ can be interpreted as the *probability* to find a parton of type *a* in a parton of type *b* with a fraction *z* of its longitudinal momentum and transverse momentum squared << μ^2.

Aside on Sudakov Form Factor

Consider the function

$$\Delta\left(\mu^2\right) \equiv \exp\left[-\int_{\mu_0^2}^{\mu^2} \frac{d\kappa^2}{\kappa^2} \int_0^1 dz \frac{\alpha_s}{2\pi} \hat{P}(z) \right],$$

which involves the unregulated version of the splitting function but, in a sense, contains the information about the regulation of the soft singularity ($z \to 1$).

This is the bare* version of the Sudakov Form Factor mentioned earlier.

Using $P(z) = \hat{P}(z)_+$ and $P(z) = 0$ outside of $0 \le z \le 1$, we can write

$$\mu^2 \frac{\partial}{\partial \mu^2} q\left(x, \mu^2\right) = \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{dz}{z} \hat{P}(z) q\left(\frac{x}{z}, \mu^2\right) + \frac{q\left(x, \mu^2\right)}{\Delta\left(\mu^2\right)} \mu^2 \frac{\partial}{\partial \mu^2} \Delta\left(\mu^2\right)$$

or

$$\mu^2 \frac{\partial}{\partial \mu^2} \left(\frac{q\left(x, \mu^2\right)}{\Delta\left(\mu^2\right)} \right) = \frac{1}{\Delta\left(\mu^2\right)} \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{dz}{z} \hat{P}(z) q\left(\frac{x}{z}, \mu^2\right)$$

* In physical applications the physics will control the soft singularity as was displayed earlier.

The solution can be written

$$q(x,\mu^2) = \Delta(\mu^2) q(x,\mu_0^2) + \int_{\mu_0^2}^{\mu^2} \frac{d\kappa^2}{\kappa^2} \frac{\Delta(\mu^2)}{\Delta(\kappa^2)} \frac{\alpha_s(\kappa^2)}{2\pi} \int_x^1 \frac{dz}{z} \hat{P}(z) q\left(\frac{x}{z}, \kappa^2\right)$$

So we interpret $\Delta(\mu^2)$ as the probability to evolve from μ_0^2 to μ^2 without splitting and $\Delta(\mu^2)/\Delta(\kappa^2)$ as the corresponding probability for the range κ^2 to μ^2.

This interpretation will be helpful when thinking about time-like evolution and parton showering (see M. Seymour)

Back to DGLAP

We really have a matrix problem ($2n_f+1$ dimensional)

$$\mu^2 \frac{\partial}{\partial \mu^2}\begin{pmatrix} q_i\left(x,\mu^2\right) \\ g\left(x,\mu^2\right)\end{pmatrix} = \frac{\alpha_s\left(\mu^2\right)}{2\pi}\sum_{q_j,\bar{q}_j}\int_x^1 \frac{d\xi}{\xi}\times$$

$$\begin{pmatrix} P_{q_iq_j}\left(\frac{x}{\xi},\alpha_s\left(\mu^2\right)\right) & P_{q_ig}\left(\frac{x}{\xi},\alpha_s\left(\mu^2\right)\right) \\ P_{gq_j}\left(\frac{x}{\xi},\alpha_s\left(\mu^2\right)\right) & P_{gg}\left(\frac{x}{\xi},\alpha_s\left(\mu^2\right)\right)\end{pmatrix}\begin{pmatrix} q_j\left(\xi,\mu^2\right) \\ g\left(\xi,\mu^2\right)\end{pmatrix}$$

Symmetries to the rescue (charge conjugation, $SU(n_f)$,…) –

- $P_{q_iq_j} = P_{\bar{q}_i\bar{q}_j}$; $P_{q_i\bar{q}_j} = P_{\bar{q}_iq_j}$
- $P_{q_ig} = P_{\bar{q}_ig} \equiv P_{qg}$; $P_{gq_i} = P_{g\bar{q}_i} \equiv P_{gq}$
- **QCD is flavor blind and, at leading order, is flavor diagonal**

Quark number and momentum conservation***

$$\int_0^1 dx P_{qq}^{(0)}(x) = 0 \quad ; \quad \int_0^1 dx\, x\left[P_{qq}^{(0)}(x) + P_{gq}^{(0)}(x)\right] = 0 \quad ;$$

$$\int_0^1 dx\, x\left[2n_f P_{qg}^{(0)}(x) + P_{gg}^{(0)}(x)\right] = 0$$

*** Verify these sum rules with forms below!

In summary

$$P_{qq}^{(0)}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]$$

$$P_{qg}^{(0)}(x) = T_R\left[x^2 + (1-x)^2\right]$$

$$P_{gq}^{(0)}(x) = C_F\left[\frac{1+(1-x)^2}{x}\right]$$

$$P_{gg}^{(0)}(x) = 2C_A\left[\frac{x}{(1-x)_+} + \frac{(1-x)}{x} + x(1-x)\right] + \delta(1-x)\frac{11C_A - 4n_f T_R}{6}$$

We can further explore the DGLAP equation by taking moments – define

$$f\left(j,\mu^2\right) = \int_0^1 dx\, x^{j-1}\, f\left(x,\mu^2\right),\ f = q_i,\ g$$

For a non-singlet quark distribution, $q_{NS} = q - \bar{q}$, with evolution controlled by P_{qq}

$$\mu^2 \frac{\partial}{\partial \mu^2} q_{NS}\left(j,\mu^2\right) = \frac{\alpha_s\left(\mu^2\right)}{2\pi}\gamma_{qq}\left(j,\alpha_s\left(\mu^2\right)\right) q_{NS}\left(j,\mu^2\right)$$

$$\gamma_{qq}\left(j,\alpha_s\left(\mu^2\right)\right) = \int_0^1 dx\, x^{j-1} P_{qq}\left(x,\alpha_s\left(\mu^2\right)\right)$$

In leading order (1-loop – no μ dependence in *P*) the solution is

$$q_{NS}\left(j,\mu^2\right)=\left(\frac{\ln\left(\mu^2/\Lambda^2_{QCD}\right)}{\ln\left(\mu_0^2/\Lambda^2_{QCD}\right)}\right)^{\frac{2\gamma^{(0)}_{qq}(j)}{\beta_0}} q_{NS}\left(j,\mu_0^2\right)$$

behavior characteristic of gauge theories where $\gamma^{(0)}_{qq}(j)$ is often called the "anomalous dimension" for the j^{th} moment.

In a similar way we can study the moments of the singlet distribution

$$\Sigma\left(x,\mu^2\right)\equiv\sum_i\left[q_i\left(x,\mu^2\right)+\overline{q}_i\left(x,\mu^2\right)\right],$$

which mixes with the gluon. The moments obey

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix}\Sigma\left(j,\mu^2\right)\\ g\left(j,\mu^2\right)\end{pmatrix}=\frac{\alpha_s\left(\mu^2\right)}{2\pi}\times$$
$$\begin{pmatrix}\gamma_{qq}\left(j,\alpha_s\left(\mu^2\right)\right) & 2n_f\gamma_{qg}\left(j,\alpha_s\left(\mu^2\right)\right)\\ \gamma_{gq}\left(j,\alpha_s\left(\mu^2\right)\right) & \gamma_{gg}\left(j,\alpha_s\left(\mu^2\right)\right)\end{pmatrix}\begin{pmatrix}\Sigma\left(j,\mu^2\right)\\ g\left(j,\mu^2\right)\end{pmatrix}$$

where*

$$\gamma^{(0)}_{qq}(j)=C_F\left[-\frac{1}{2}+\frac{1}{j(j+1)}-2\sum_{k=2}^{j}\frac{1}{k}\right]$$

*You should check these!

$$\gamma_{qg}^{(0)}(j) = T_R\left[\frac{2+j+j^2}{j(j+1)(j+2)}\right]$$

$$\gamma_{gq}^{(0)}(j) = C_F\left[\frac{2+j+j^2}{j(j^2-1)}\right]$$

$$\gamma_{gg}^{(0)}(j) = 2C_A\left[-\frac{1}{12}+\frac{1}{j(j-1)}+\frac{1}{(j+1)(j+2)}-\sum_{k=2}^{j}\frac{1}{k}\right]-\left(\frac{2}{3}\right)n_f T_R$$

These moments can be inverted with the inverse Mellin transformation (at least numerically).

We will be satisfied here with a few simple observations.

- **Since $d_{qq}(1)=0$, the number of valence quarks does not evolve – flavor is conserved by QCD**

- **Since $d_{qq}(j\geq 2)<0$, the non-singlet quark distribution evolves by decreasing at large *x* and increasing at small *x* – *as expected as the quark emits gluons***

Next note that

$$\int_0^1 dx\, x^{j-1}\frac{1}{x} = \frac{1}{j-1}$$

$$\int dx\, x^{j-1}\frac{1}{(1-x)_+} = -\int_0^1 dx\,\frac{x^{j-1}-1}{x-1} \sim \ln j \quad j>>1$$

The pole at *j=1* means the fixed order analysis is unreliable for the limit of small x (see J. Collins)**.**

The large x behavior can be inferred from the ln *j* behavior of the anomalous dimensions and the fact that

$$f\left(x,\mu^2\right)\xrightarrow[x\to 1]{}(1-x)^{a\left(\mu^2\right)} \Rightarrow f\left(j,\mu^2\right)\xrightarrow[j>>1]{} j^{-a\left(\mu^2\right)}$$

Thus we find

$$\gamma_{qq}^{(0)}\left(j>>1\right)\sim -4C_F \ln j \quad \Rightarrow$$

$$j^{-a\left(\mu_0^2\right)\left(\frac{\ln\left(\mu^2/\Lambda_{QCD}^2\right)}{\ln\left(\mu_0^2/\Lambda_{QCD}^2\right)}\right)^{-4C_F \ln j/\beta_0}} = j^{-a\left(\mu_0^2\right)-4C_F \ln\left(\ln\left(\mu^2/\Lambda_{QCD}^2\right)/\ln\left(\mu_0^2/\Lambda_{QCD}^2\right)\right)/\beta_0}$$

$$\Rightarrow q_{NS}\left(x,\mu^2\right)\sim (1-x)^{a\left(\mu_0^2\right)+4C_F \ln\left(\ln\left(\mu^2/\Lambda_{QCD}^2\right)/\ln\left(\mu_0^2/\Lambda_{QCD}^2\right)\right)/\beta_0}$$

For the singlet plus gluon system we must find the corresponding eigenvalues and eigenvectors. With j=2, the momentum sum integral, we have

Eigenvector	Eigenvalue
$\Sigma(2,\mu^2)+g(2,\mu^2)$	0
$\Sigma(2,\mu^2)-(n_f/4C_F)\, g(2,\mu^2)$	$-4C_F/3 - n_f/3$

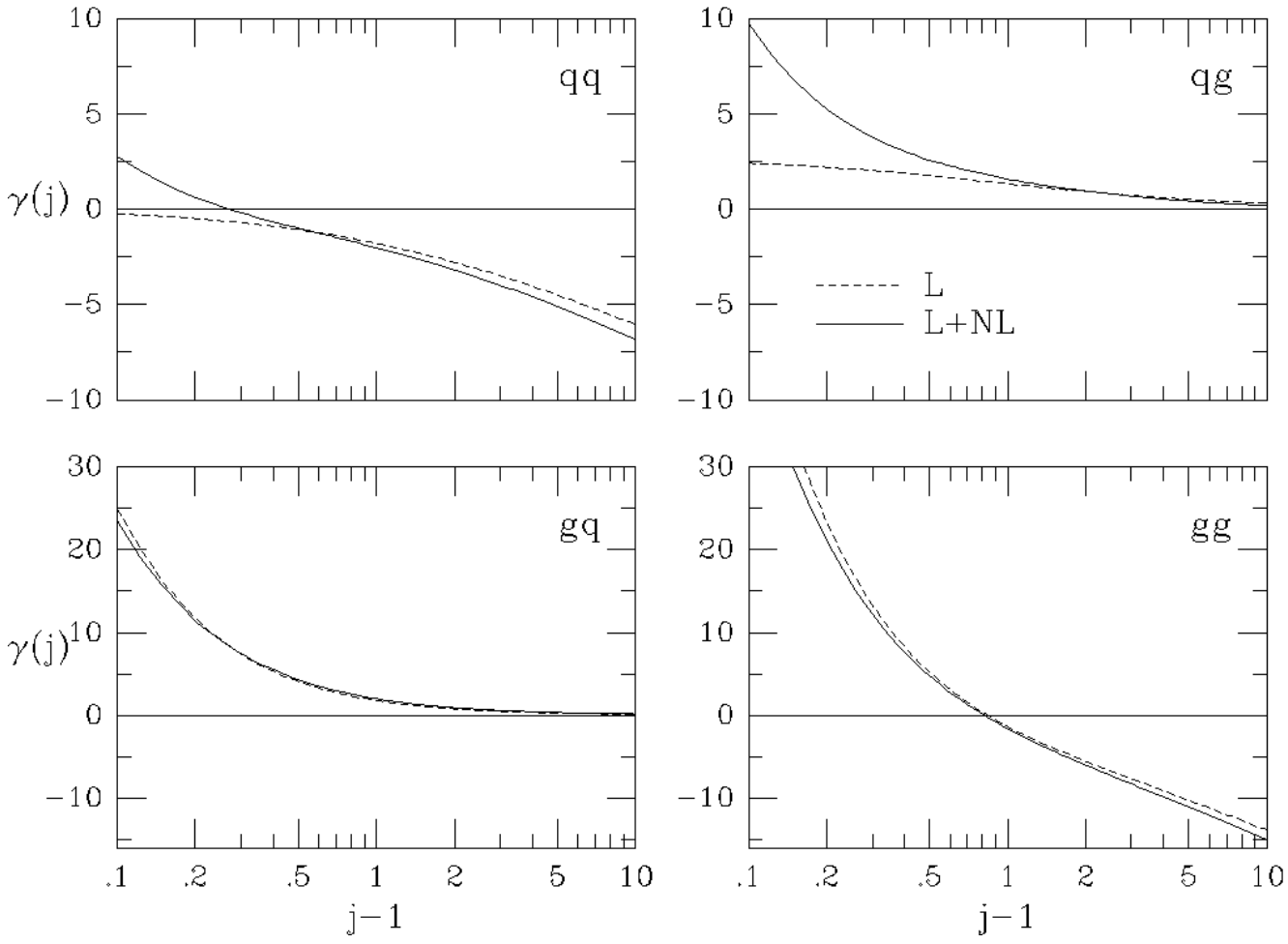
- **The first line confirms that total momentum is conserved during evolution!!**
- **Since the second eigenvalue is < 0, the second eigenvector vanishes asymptotically (ln $\mu^2\to$infinity)**

$$\Sigma\left(2,\mu^2\right)-\frac{n_f}{4C_f}g\left(2,\mu^2\right)\xrightarrow[\ln\mu^2\to\infty]{}0 \Rightarrow \frac{\Sigma\left(2,\mu^2\right)}{g\left(2,\mu^2\right)}\xrightarrow[\ln\mu^2\to\infty]{}\frac{n_f}{4C_f}=\frac{3n_f}{16}$$

Hence the (truly!) asymptotic momentum ratios (at leading order) are

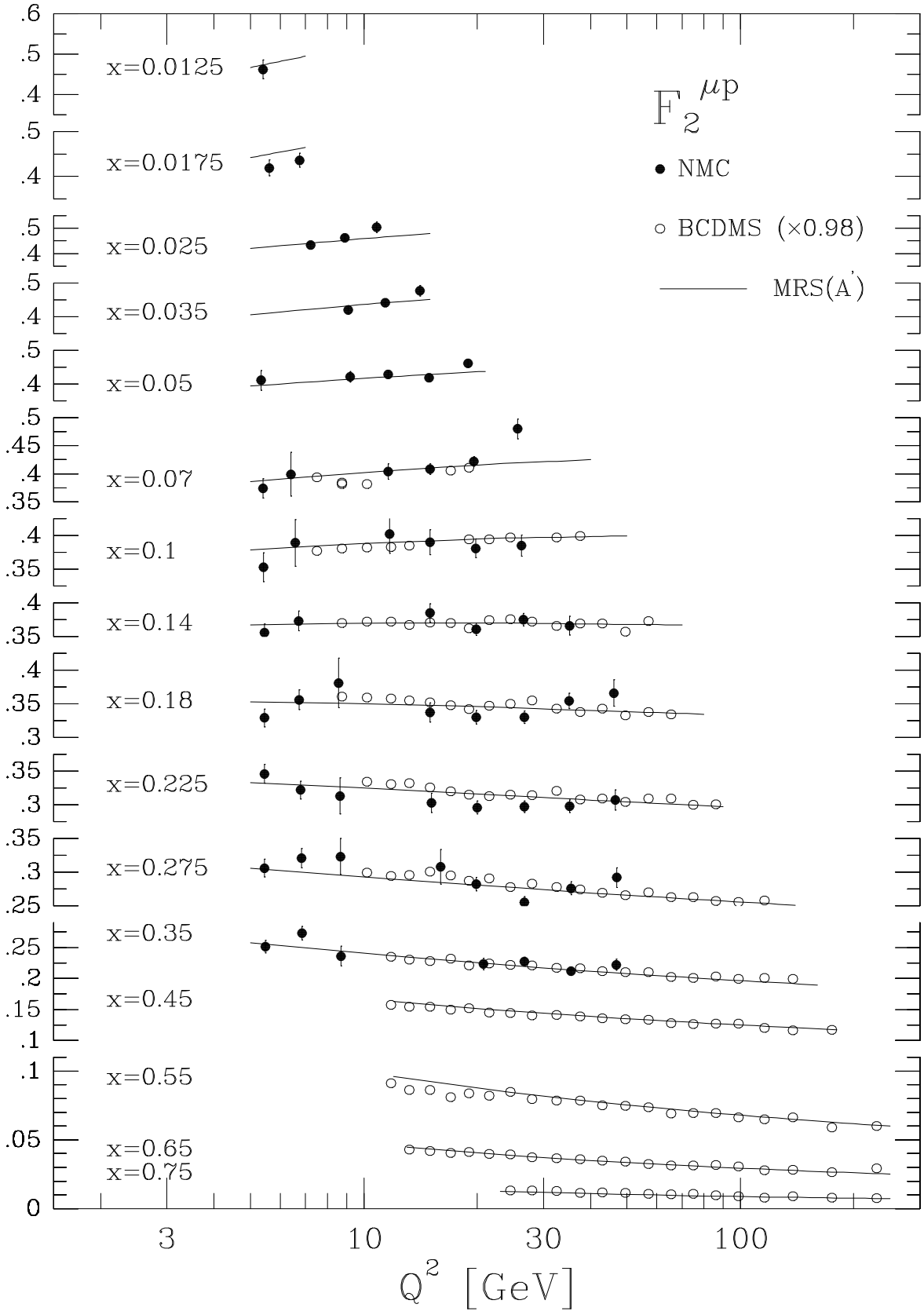
$$f_q = \frac{3n_f}{16+3n_f} \xrightarrow[n_f=6]{} 53\% \qquad f_g = \frac{16}{16+3n_f} \xrightarrow[n_f=6]{} 47\%$$

Numerically the anomalous dimensions look like –



Thus we expect that the distributions shrink at large *x* and grow at small *x* as μ^2 increases.

The expected violations of scaling are seen experimentally-



Status – Parton Model + pQCD

- Basic structure of parton model remains valid
- QCD coupling is small at short distance, large at large distance
- Can factor the complicated (hard to calculate) long distance, confining behavior from the short distance perturbative behavior
- Fix the long distance behavior experimentally
- Perturbation theory predicts the short distance evolution
- Can eliminate the dependence on the long distance physics with the proper choice of experimental quantity