Lepton Pair, W, and Z Production

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Lecture I

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   - history, motivation, and early ideas

2. Simple calculation
   - kinematics, matrix element, and phase space
   - features and shortcomings

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   - $\frac{d\sigma}{dQ^2}$

Lecture II

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2. W and Z production
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4. W production
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   - charged lepton distributions
   - roles in global fits and W mass measurements
5. Conclusions
6. References

Lepton Pair Production

S.D. Drell and T.-M. Yan, Phys. Rev. Lett. **25**, 316 (1970)

- electromagnetic probe of a hadron-hadron process

- compare to

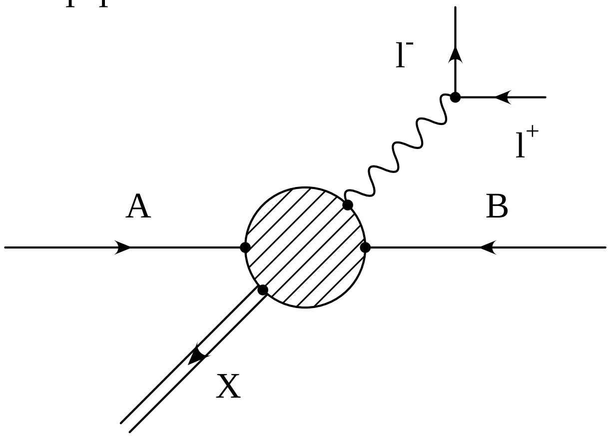
  - DIS: E-M probe of a single hadron process

  - e^+e^-: E-M probe of hadron production

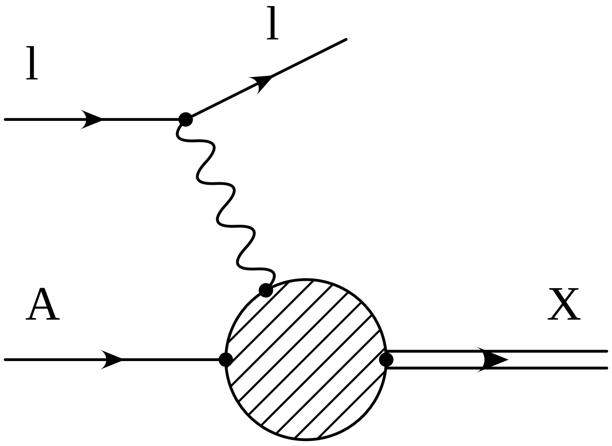
- simple description in terms of the (then new) parton model

- mass of the pair could be varied to insure that the parton momentum fractions were neither too *small* nor too *large* (avoid problems with x near 0 or 1)

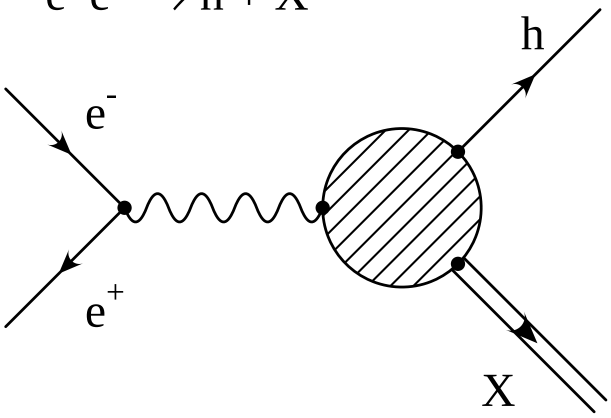
$l^+ l^-$



DIS



$e^- e^+ \rightarrow h + X$



- structure of the parton model calculation *preserved* in the presence of QCD corrections

- first example of a *calculable* hadron-hadron process in the context of the parton model

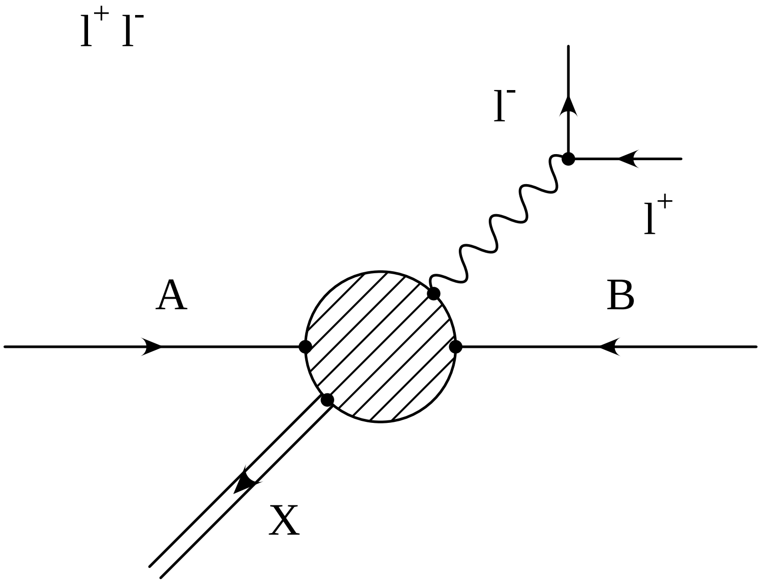
- process is of historical interest (2 Nobel prizes)

- pedagogical importance - one of the early calculations of higher order QCD corrections

- important for precision Standard Model measurements

- important roles in searches for new physics

Basic Idea



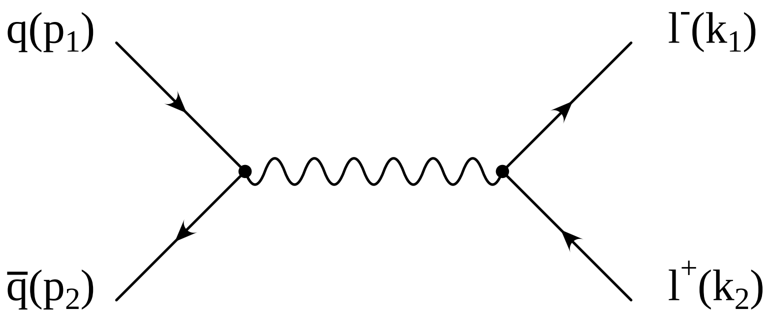
Producing a virtual photon with mass Q and $Q^2 > 0$

Our task is to figure out what is in the shaded circle in the figure

Simplest possibility: $q\bar{q} \to l^+ l^-$

Represents purely E-M process in the context of the parton model (treating the quarks as free)

Born Term



Lorentz invariant variables

$$
\begin{aligned}
\hat{s} &= (p_1 + p_2)^2 = (k_1 + k_2)^2 \\
\hat{t} &= (p_1 - k_1)^2 = (p_2 - k_2)^2 \\
\hat{u} &= (p_1 - k_2)^2 = (p_2 - k_1)^2
\end{aligned}
$$

Matrix element

$$M = e_q \frac{e^2}{\hat{s}} \overline{u}(k_1) \gamma_\mu v(k_2) \overline{v}(p_2) \gamma^\mu u(p_1)$$

Spin/color averaged matrix element squared

$$\overline{\sum}|M|^2 = \frac{e_q^2 e^4}{\hat{s}^2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)3\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$$
$$\mathrm{Tr}\left[\not{p}_1\gamma^\nu\not{p}_2\gamma^\mu\right]\mathrm{Tr}\left[\not{k}_2\gamma_\nu\not{k}_1\gamma_\mu\right]$$

$$= \frac{4}{3}\frac{e_q^2 e^4}{\hat{s}^2}\left[p_1^\nu p_2^\mu + p_1^\mu p_2^\nu - g^{\mu\nu} p_1 \cdot p_2\right]$$
$$\left[k_{2\nu}k_{1\mu} + k_{2\mu}k_{1\nu} - g_{\mu\nu} k_1 \cdot k_2\right]$$

Red factors are for the spin average

Blue factors are for the color average

Forming the indicated dot products yields

$$\overline{\sum}|M|^2 = \frac{4}{3}\frac{e_q^2 e^4}{\hat{s}^2}\left[2p_1 \cdot k_2 p_2 \cdot k_1 + 2p_1 \cdot k_1 p_2 \cdot k_2\right]$$
$$= \frac{2}{3}\frac{e_q^2 e^4}{\hat{s}^2}\left[\hat{t}^2 + \hat{u}^2\right]$$

Center of mass frame: the 4-vectors are

$$
\begin{aligned}
p_1 &= \frac{\sqrt{\hat{s}}}{2}(1,0,0,1) \\
p_2 &= \frac{\sqrt{\hat{s}}}{2}(1,0,0,-1) \\
k_1 &= \frac{\sqrt{\hat{s}}}{2}(1,\sin(\theta),0,\cos(\theta)) \\
k_2 &= \frac{\sqrt{\hat{s}}}{2}(1,-\sin(\theta),0,-\cos(\theta))
\end{aligned}
$$

yielding the Lorentz scalars

$$
\begin{aligned}
\hat{t} &= -\frac{\hat{s}}{2}(1-\cos(\theta)) \\
\hat{u} &= -\frac{\hat{s}}{2}(1+\cos(\theta))
\end{aligned}
$$

with

$$\hat{t}^2+\hat{u}^2 = \frac{s^2}{2}(1+\cos^2(\theta))$$

Inserting these relations into our result yields the answer we seek:

$$\overline{\sum}|M|^2 = \frac{e_q^2 e^4}{3}(1+\cos^2(\theta))$$

To make use of this result we need to convert it to a cross section. For this we need the two-body Lorentz invariant phase space factor:

$$
\begin{aligned}
PS^{(2)} &= \frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2} \\
&\quad \times (2\pi)^4\delta^4(p_1+p_2-k_1-k_2) \\
&= \frac{d^3k}{16\pi^2 E_1E_2}\delta(\sqrt{\hat{s}}-E_1-E_2).
\end{aligned}
$$

In the center-of-momentum frame we have $k = |\vec{k}_1| = |\vec{k}_2|$ so that in this frame we can write

$$
\begin{aligned}
d(E_1+E_2) &= d\sqrt{\hat{s}} \\
&= kdk\left(\frac{1}{E_1}+\frac{1}{E_2}\right) \\
&= kdk\frac{E_1+E_2}{E_1E_2}.
\end{aligned}
$$

with $k = \sqrt{\hat{s}}/2$.

This allows the phase space factor to be written as

$$
\begin{aligned}
PS^{(2)} &= \frac{k^2dkd\Omega}{16\pi^2E_1E_2}\delta(\sqrt{\hat{s}}-E_1-E_2) \\
&= \frac{kd\sqrt{\hat{s}}d\Omega}{16\pi^2\sqrt{\hat{s}}}\delta(\sqrt{\hat{s}}-E_1-E_2) \\
&= \frac{d\Omega}{32\pi^2} \\
&= \frac{d\cos(\theta)}{16\pi}
\end{aligned}
$$

To get a cross section we multiply the phase space factor times the spin and color averaged squared matrix element and multiply that by a flux factor of $1/2\hat{s}$ yielding

$$\begin{aligned}\sigma(q\overline{q} \to l^+l^-) &= \frac{1}{2\hat{s}} \int_{-1}^{1} \frac{d\cos(\theta)}{16\pi} \frac{e_q^2 e^4}{3}(1 + \cos^2(\theta)) \\ &= \frac{e_q^2}{3} \frac{(4\pi\alpha)^2}{16\pi} \frac{1}{2\hat{s}} \frac{8}{3}\end{aligned}$$

where the fine structure constant $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$.

The final result for the parton-level cross section is

$$\sigma(q\overline{q} \to l^+l^-) = \frac{4\pi\alpha^2}{9\hat{s}} e_q^2 \equiv \sigma_0.$$

Hadronic Cross Section

Introduce the parton distribution functions: let $q_A(x)dx$ denote the probability of finding a parton of flavor q in hadron A with a momentum fraction lying between x and $x + dx$. Convolute the parton-level cross section σ_0 with the appropriate quark and antiquark parton distribution functions:

$$\sigma(AB \to l^+l^- + X) = \sum_q \int dx_a dx_b \, \sigma_0 \left[q(x_a)\overline{q}(x_b) + a \leftrightarrow b\right]$$

Note: remember to symmetrize with respect to the beam and target particles. This corresponds to $\hat{t} \leftrightarrow \hat{u}$ here, so σ_0 is unchanged.

Differential Distributions

The total cross section involves a convolution with products of parton distributions. In order to test the theory or to learn more about the parton distributions it has proven to be convenient to undo one or both of the integrations by looking a differential distributions. If we ignore external hadronic masses, we can relate the hadronic and partonic center of mass energies as follows:

$$s = (p_A + p_B)^2 = 2p_A \cdot p_B = \frac{2p_1 \cdot p_2}{x_a x_b} = \frac{\hat{s}}{x_a x_b}$$

where it has been assumed that $p_1 = x_a p_A$ and $p_2 = x_b p_B$.

Lepton pair mass distribution

For $q\bar{q} \to l^+ l^-$ the invariant mass of the lepton pair is just $Q^2 = \hat{s}$. Thus,

$$\frac{d\sigma}{dQ^2} = \sum_q \int dx_a dx_b \, H_q(x_a, x_b) \, \sigma_0 \, \delta(Q^2 - \hat{s})$$

Here the sum over the products of parton distributions is denoted by the function $H_q(x_a, x_b)$. Next, evaluate the δ function as follows:

$$\int dx_a dx_b \delta(Q^2 - x_a x_b s) = \int \frac{dx_a}{x_a s} \delta(x_b - Q^2/x_a s).$$

Thus,

$$\frac{d\sigma}{dQ^2} = \sum_q \int \frac{dx_a}{x_a s} H_q(x_a, \frac{Q^2}{x_a s}) \frac{4\pi\alpha^2}{9Q^2} e_q^2.$$

Scaling

It is convenient to define a dimensionless parameter $\tau = Q^2/s = x_a x_b$ so that $0 \le \tau \le 1$. Then, we can write

$$Q^4 \frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{9} \sum_q \int_\tau^1 \frac{dx_a}{x_a} \; \tau H_q(x_a, \frac{\tau}{x_a}) e_q^2$$

Note: $Q^4 \frac{d\sigma}{dQ^2}$ is dimensionless

Righthand side is a function only of τ at this level of approximation

Plot $Q^4 \frac{d\sigma}{dQ^2}$ or $(Q^3 \frac{d\sigma}{dQ})$ for different values of s

Should lie on a universal "scaling" curve

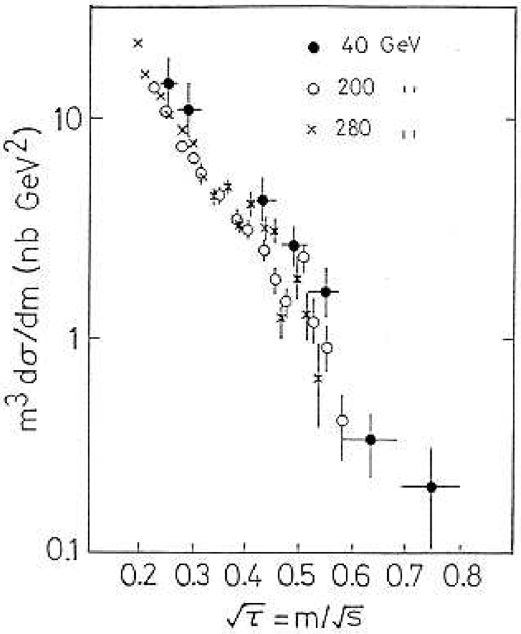


Fig. 5.13. Approximate scaling of $m^3 d\sigma/dm$ for Drell-Yan pair production in $\pi^- p$ scattering with lab momentum from 40 to 280 GeV [Phys. Lett. 96B, 417 (1980)].

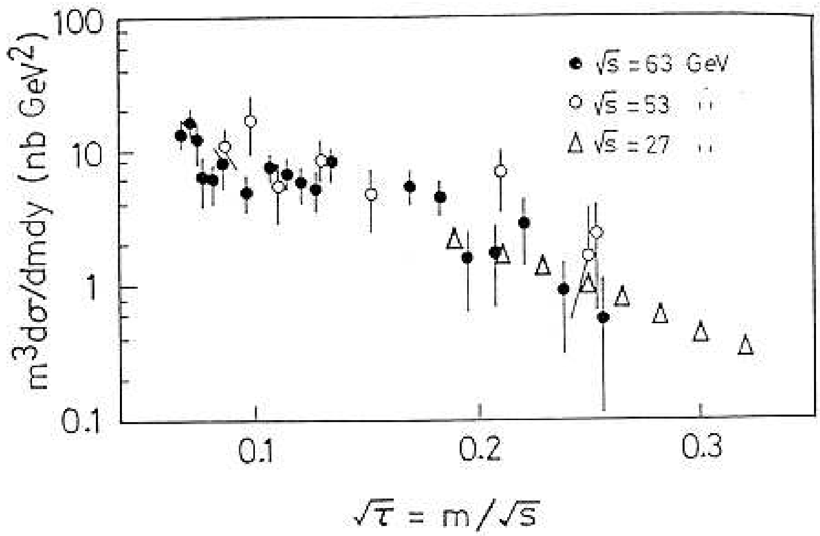


Fig. 5.14. Approximate scaling of $m^3 d\sigma/dm\, dy(y = 0)$ for Drell-Yan pair production in pp scattering [Phys. Lett. 91B, 475 (1980)].

Longitudinal Momentum Distributions

Define a longitudinal scaling variable

$$x_F = p_l/p_{lmax} \approx 2p_l/\sqrt{s}$$

where p_l is the longitudinal momentum in the hadron-hadron cms. For the $q\bar{q} \to l^+l^-$ subprocess we are interested in the longitudinal momentum of the lepton pair. We have

$$p_1 = x_a\tfrac{\sqrt{s}}{2}(1,0,0,1) \quad p_2 = x_b\tfrac{\sqrt{s}}{2}(1,0,0,-1)$$
$$E = \tfrac{\sqrt{s}}{2}(x_a + x_b) \quad p_l = \tfrac{\sqrt{s}}{2}(x_a - x_b)$$

which yields $x_F = x_a - x_b$.

One can use this to define a double differential cross section

$$\frac{d\sigma}{dQ^2 dx_F} = \frac{4\pi\alpha^2}{9Q^4}\sum_q e_q^2 \int_\tau^1 \frac{dx_a}{x_a}\ \tau H_q(x_a, \frac{\tau}{x_a})\delta(x_F - x_a + \frac{\tau}{x_a})$$

The δ function constraint can be solved for x_a yielding

$$x_a = \frac{1}{2}\left(x_F + \sqrt{x_F^2 + 4\tau}\right).$$

Using $x_b = \tau/x_a$ one derives

$$x_b = \frac{1}{2}\left(-x_F + \sqrt{x_F^2 + 4\tau}\right).$$

The Jacobian factor from the δ function introduces a factor of $x_a/(x_a+x_b)$ so the final result can be written as

$$\frac{d\sigma}{dQ^2 dx_F} = \frac{4\pi\alpha^2}{9Q^4}\frac{1}{\sqrt{x_F^2+4\tau}}\tau\sum_q e_q^2 H_q(x_a, \frac{\tau}{x_a}).$$

Rapidity

The rapidity variable is defined as

$$y = \frac{1}{2}\ln\frac{E+p_l}{E-p_l} = \frac{1}{2}\ln x_a x_b$$

which yields

$$x_a = \sqrt{\tau}e^y \text{ and } x_b = \sqrt{\tau}e^{-y}.$$

Changing variables from (Q^2, x_F) to (y, τ) is done using

$$dQ^2 dx_F = dy d\tau\ s\sqrt{x_F^2+4\tau}$$

which gives

$$\frac{d\sigma}{dy d\tau} = \frac{4\pi\alpha^2}{9s}\sum_q \frac{e_q^2}{\tau}H_q(x_a, x_b)$$

with $x_{a,b}$ given above.

QCD

To this point we have been reviewing simple parton model results for lepton pair production. Where does QCD enter and how does it modify what we've done so far?

Easy to answer – harder to prove:

To leading-logarithm accuracy all of the foregoing expressions are correct provided that we use scale dependent parton distributions $q_a(x_a, Q^2)$, etc. when evaluating the function $H_q(x_a, x_b)$! These are obtained as solutions of the DGLAP equations.

We shall examine the $\mathcal{O}(\alpha_s)$ corrections to lepton pair production shortly in order to see how this comes about. For now, it is sufficient to note that the preceeding results pick up a logarithmic dependence on Q^2 which modifies the scaling arguments presented earlier.

p_T Distribution

- To the order we are working for the hard scattering subprocess , *i.e.*, $\mathcal{O}(\alpha_s^0)$, no transverse momentum is generated via the subprocess itself ($q\overline{q} \to l^+l^-$).

- Transverse momenta associated with initial state gluon emission have been integrated out in the process of solving the DGLAP evolution equations for the scale dependent pdf's.

- The pdf's retain their dependence on the longitudinal momentum fractions – in the preceeding discussion these have been fixed by specifying Q^2 and either x_F or the rapidity.

- Early parton model predictions treated the lepton pair transverse momentum distributions by attributing a gaussian transverse momentum distribution to the incoming partons (“intrinsic k_T”)
  - Data showed $< k_T > \simeq$ 760 MeV per parton at p_{lab} =400 GeV ($\sqrt{s} = 27.4$ GeV)
  - $< k_T >$ larger than expectations based on hadron size.
  - Data showed a non-gaussian tail.

- Above observations suggested that there was more than just the simple parton model at work here. Turn next to examining higher order QCD corrections to the description obtained thus far.

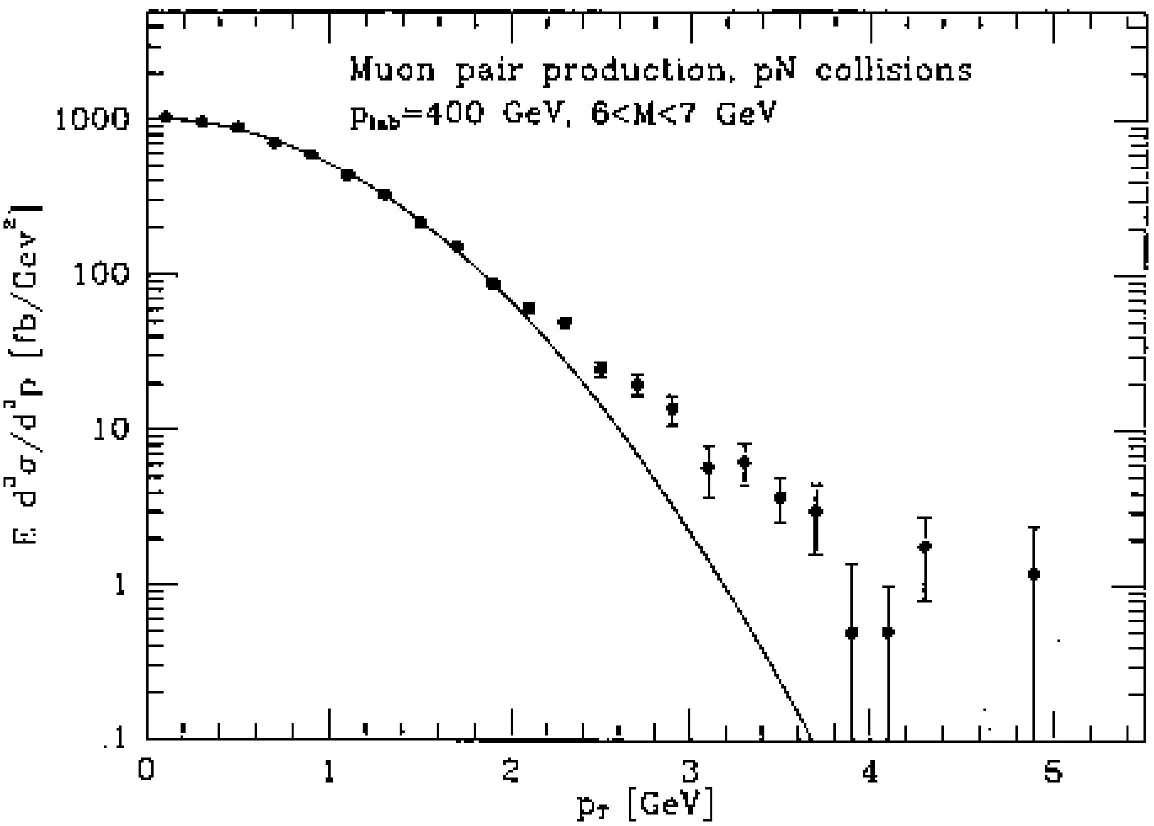
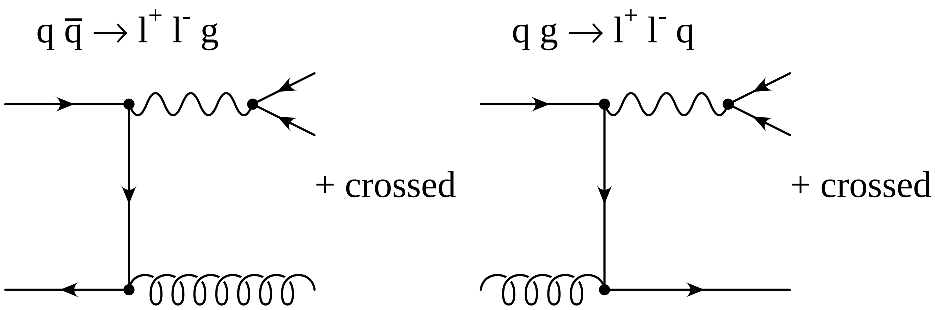


Fig. 9.2. The lepton pair transverse momentum from the CFS collaboration [4]. The curve corresponds to a Gaussian intrinsic k_T distribution for the annihilating partons.

Figure from "QCD and Collider Physics" by Ellis, Stirling, and Webber. Note the gaussian-like behavior in the low-p_T region with a non-gaussian tail appearing at higher values of p_T.

$\mathcal{O}(\alpha_s)$ QCD Contributions



- Lepton pair recoils against a quark or gluon
- Can simplify the calculation by considering the production of a virtual photon of mass Q. Label the momenta by $q(p_1) + \overline{q}(p_2) \to \gamma^*(q) + g(k_3)$.

$$\begin{aligned} d\sigma &= \frac{1}{2\hat{s}} \overline{\sum} |M(q\overline{q} \to \gamma^* g)|^2 \frac{d^3 q}{(2\pi)^3 2E_q} \frac{d^3 k_3}{(2\pi)^3 2E_3} \\ &\quad \times (2\pi)^4 \delta(p_1 + p_2 - q - k_3) \\ &= \frac{1}{2\hat{s}} \overline{\sum} |M(q\overline{q} \to \gamma^* g)|^2 \frac{\hat{s} - Q^2}{2\hat{s}} \frac{d\Omega}{16\pi^2} \end{aligned}$$

with

$$\begin{aligned} q &= \left[\frac{\hat{s} + Q^2}{2\sqrt{\hat{s}}}, \frac{\hat{s} - Q^2}{2\sqrt{\hat{s}}} \sin(\theta), 0, \frac{\hat{s} - Q^2}{2\sqrt{\hat{s}}} \cos(\theta) \right] \\ p_1 &= \frac{\sqrt{\hat{s}}}{2}(1, 0, 0, 1) \end{aligned}$$

so that

$$\hat{t}, \hat{u} = -\frac{\hat{s} - Q^2}{2}(1 \mp \cos(\theta)) \quad \text{and} \quad d\cos(\theta) = \frac{2}{\hat{s} - Q^2} d\hat{t}$$

Therefore, we have

$$\frac{d\sigma}{d\hat{t}} = \frac{1}{16\pi\hat{s}^2} \overline{\sum} |M(q\overline{q} \to \gamma^* g)|^2$$

Next, consider the decay process $\gamma^*(q) \to l^-(k_1) + l^+(k_2)$. We have $M_\mu = e\overline{u}(k_1)\gamma_\mu v(k_2)$

which yields

$$\begin{aligned}\overline{\sum}|M|^2 &= -\frac{1}{3}e^2 \mathrm{Tr}\,[\not{k}_2\gamma_\mu\not{k}_1\gamma^\mu] \\ &= \frac{2}{3}e^2\mathrm{Tr}\,[\not{k}_1\not{k}_2] = \frac{16\pi\alpha Q^2}{3}\end{aligned}$$

Next, consider the full $2 \to 3$ subprocess

$$\begin{aligned}d\sigma &= \frac{1}{2\hat{s}}\overline{\sum}|M(q\overline{q} \to l^+l^-g)|^2 \frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2}\frac{d^3k_3}{(2\pi)^3 2E_3} \\ &\quad \times (2\pi)^4\delta^4(p_1 + p_2 - k_1 - k_2 - k_3).\end{aligned}$$

Now, insert $d^4q\delta^4(q - k_1 - k_2)$ and split the matrix element into production and decay processes.

$$\begin{aligned}d\sigma &= \frac{1}{2\hat{s}}\overline{\sum}|M(q\overline{q} \to \gamma^* g)|^2 \frac{d^3k_3}{(2\pi)^3 2E_3} d^4q \\ &\quad \times \delta^4(p_1 + p_2 - q - k_3)\frac{1}{Q^4}\overline{\sum}|M(\gamma^* \to l^+l^-)|^2 \\ &\quad \times \frac{d^3k_1}{(2\pi)^3 2E_1}\frac{d^3k_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^4(q - k_1 - k_2)\end{aligned}$$

The factors on the last line are just those for two-body phase space, so they can be rewritten as $\frac{d\Omega}{32\pi^2}$, thereby simplifying the result to

$$d\sigma = \frac{1}{16\pi\hat{s}Q^4}\overline{\sum}|M(q\overline{q} \to \gamma^* g)|^2 \frac{d^3k_3}{(2\pi)^3 2E_3}\frac{16\pi\alpha Q^2}{3}$$

Next rewrite the k_3 differential as

$$\frac{d^3k_3}{(2\pi)^3 2E_3} = \frac{k_3 dk_3 d\cos(\theta)}{8\pi^2} = \frac{d\hat{t}dQ^2}{16\pi^2\hat{s}} \quad \text{with} \quad k_3 = \frac{\hat{s}-Q^2}{2\sqrt{\hat{s}}}.$$

Therefore, one obtains

$$\frac{d\sigma}{dQ^2d\hat{t}} = \frac{1}{16\pi^2\hat{s}^2}\frac{\alpha}{3Q^2}\overline{\sum}|M(q\bar{q}\to\gamma^* g)|^2.$$

The end result is that the $2\to 3$ cross section is proportional to a simpler $2\to 2$ cross section:

$$\frac{d\sigma}{dQ^2d\hat{t}}(q\bar{q}\to l^+l^-g) = \frac{\alpha}{3\pi Q^2}\frac{d\sigma}{d\hat{t}}(q\bar{q}\to\gamma^* g).$$

The two-body cross sections are easily calculated in the conventional fashion which, together with the relation just derived, yield the following results:

$$\begin{aligned}\frac{d\sigma}{dQ^2d\hat{t}}(q\bar{q}\to l^+l^-g) &= \frac{\alpha^2\alpha_s e_q^2}{Q^2\hat{s}^2}\frac{8}{27}\left[\frac{\hat{t}}{\hat{u}}+\frac{\hat{u}}{\hat{t}}+\frac{2Q^2\hat{s}}{\hat{t}\hat{u}}\right]\\ \frac{d\sigma}{dQ^2d\hat{t}}(qg\to l^+l^-q) &= -\frac{\alpha^2\alpha_s e_q^2}{Q^2\hat{s}^2}\frac{1}{9}\left[\frac{\hat{t}}{\hat{s}}+\frac{\hat{s}}{\hat{t}}+\frac{2Q^2\hat{u}}{\hat{s}\hat{t}}\right]\end{aligned}$$

The next step is to convert these expressions to those for hadronic cross sections.

First, convolute with the appropriate parton distributions:

$$\frac{d\sigma}{dQ^2} = \sum_{ab} \int dx_a dx_b \; G_{a/A}(x_a, Q^2) G_{b/B}(x_b, Q^2) \; \frac{d\sigma_{ab}}{dQ^2 d\hat{t}} \; d\hat{t}$$

Next, use the relations

$$p_T^2 = \frac{\hat{t}\hat{u}}{\hat{s}} \quad \text{and} \quad dp_T^2 = \frac{d\hat{t}}{\hat{s}} |\hat{u} - \hat{t}|$$

to get

$$\frac{d\sigma}{dQ^2 dp_T^2} = \sum_{ab} \int dx_a dx_b \; G_{a/A}(x_a, Q^2) G_{b/B}(x_b, Q^2) \frac{d\sigma_{ab}}{dQ^2 d\hat{t}} \frac{\hat{s}}{|\hat{u} - \hat{t}|}$$

This expression is not really that useful since normally one observes the lepton pair only in a restricted range of rapidity. What is really needed is a triple differential cross section

$$\frac{d\sigma}{dQ^2 dy dp_T^2}$$

Note: the more complicated phase space structure has resulted in a double differential cross section which still contains integrations on x_a and x_b. Now we have to undo one of those integrations.

The notation now becomes a bit tedious, for which I apologize...

Hadron-hadron rest frame

$$
\begin{aligned}
q &= \left[\sqrt{Q^2+p_T^2}\cosh y, p_T, 0, \sqrt{Q^2+p_T^2}\sinh y\right] \\
t &= (p_A-q)^2 = Q^2 - \sqrt{s}\sqrt{Q^2+p_T^2}\ e^{-y} \\
u &= (p_B-q)^2 = Q^2 - \sqrt{s}\sqrt{Q^2+p_T^2}\ e^{y}
\end{aligned}
$$

Now,

$$
\begin{aligned}
\hat{t} &= (p_1-q)^2 \Rightarrow \hat{t}-Q^2 = x_a(t-Q^2) \\
\hat{u} &= (p_2-q)^2 \Rightarrow \hat{u}-Q^2 = x_b(u-Q^2)
\end{aligned}
$$

and $\hat{s}+\hat{t}+\hat{u}=Q^2$, so

$$
x_a x_b s + x_a(t-Q^2) + x_b(u-Q^2) + Q^2 = 0
$$

which can be rewritten as

$$
x_b = -\frac{Q^2+x_a(t-Q^2)}{x_a s + (u-Q^2)}.
$$

Next, let

$$
\begin{aligned}
x_1 &= -(u-Q^2)/s = \sqrt{Q^2+p_T^2}\ e^{y}/\sqrt{s} \\
x_2 &= -(t-Q^2)/s = \sqrt{Q^2+p_T^2}\ e^{-y}/\sqrt{s}.
\end{aligned}
$$

Then, $x_b = \frac{x_a x_2 - \tau}{x_a - x_1}$ with Q^2, y, p_T^2 fixed.

Using the previously defined variables, one can get

$$\begin{aligned}
\hat{t} &= Q^2 - x_a\sqrt{s}\sqrt{Q^2 + p_T^2}\ e^{-y} \\
x_b &= \frac{x_a\sqrt{Q^2 + p_T^2}\ e^{-y}/\sqrt{s} - \tau}{x_a - \sqrt{Q^2 + p_T^2}\ e^{y}/\sqrt{s}}.
\end{aligned}$$

One can use these equations to show that

$$dx_b\, d\hat{t} = dy\, dp_T^2 \frac{x_a x_b}{x_a - x_1}.$$

Therefore,

$$\frac{d\sigma}{dQ^2 dy dp_T^2} = \sum_{ab} \int_{x_{amin}}^{1} dx_a \frac{x_a x_b}{x_a - x_1} G_{a/A}(x_a, Q^2) G_{b/B}(x_b, Q^2) \frac{d\sigma_{ab}}{dQ^2 d\hat{t}}$$

where x_{amin} is determined by $x_b = 1$, yielding

$$x_{amin} = \frac{x_1 - \tau}{1 - x_2}.$$

Consider the $q\bar{q} \to l^+ l^- g$ subprocess.

$$\frac{d\sigma}{dQ^2 dy dp_T^2} = \frac{\alpha^2 \alpha_s}{Q^2} \frac{8}{27} \int_{x_{amin}}^{1} dx_a \frac{x_a x_b}{x_a - x_1} \sum_q H_q(x_a, x_b, Q^2) \times \frac{1}{\hat{s}^2} \, \frac{\hat{t}^2 + \hat{u}^2 + 2Q^2 \hat{s}}{\hat{t}\hat{u}}$$

Using variables defined previously, as well as $x_T = 2p_T/\sqrt{s}$, this expression can be rewritten as

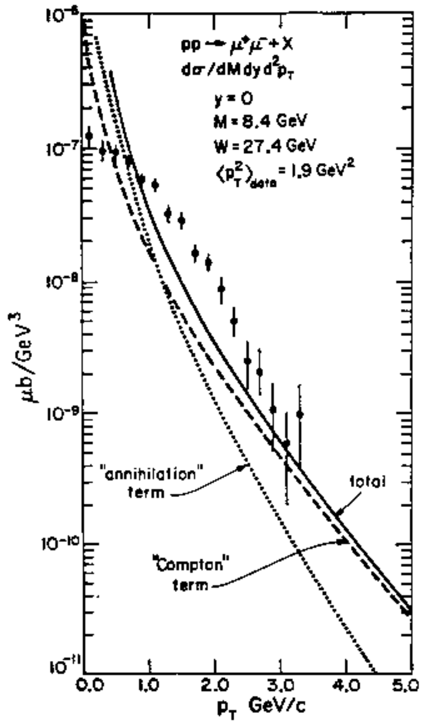
$$\frac{d\sigma}{dQ^2 dy dp_T^2} = \frac{\alpha^2 \alpha_s}{sQ^2} \frac{8}{27} \frac{1}{p_T^2} \int_{x_{amin}}^{1} dx_a \frac{x_a x_b}{x_a - x_1} \sum_q H_q(x_a, x_b, Q^2) \times \left[1 - \frac{x_T^2}{2x_a x_b} + \left(\frac{\tau}{x_a x_b} \right)^2 \right]$$

Comments

1. The annihilation term gives a p_T^{-2} tail to the p_T distribution (this falls off more s-lowly than a gaussian).

2. The Compton contribution ($qg \to l^+ l^- q$) is slightly more complicated (must include qg and gq) but similar. This actually dominates at high-p_T for pp or pN collisions.

Lesson: the tail of the p_T distribution can be calculated in QCD.



From "Applications of Perturbative QCD" by R.D. Field

If we can calculate at least the tail of the p_T distribution, then one would think that we could integrate over p_T and get an $\mathcal{O}(\alpha_s)$ contribution to the total cross section. What happens if we integrate the preceeding expression over all p_T?

1. There is a divergence as $p_T \to 0$

2. As $p_T \to 0, x_1 \to x_1^0 = \sqrt{\tau} e^y$. At the same time $x_{amin} \to x_1^0$. Therefore, the $1/(x_a - x_1)$ term contributes to the divergence at the low end of the integration range.

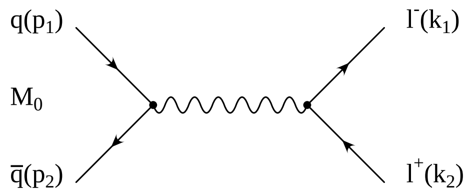
$$
\begin{aligned}
-\ln(x_{amin} - x_1) &= -\ln\left(\frac{x_1 - \tau}{1 - x_2} - x_1\right) = -\ln\frac{x_1 x_2 - \tau}{1 - x_2} \\
x_1 x_2 &= \frac{Q^2 + p_T^2}{s} = \tau + p_T^2/s
\end{aligned}
$$

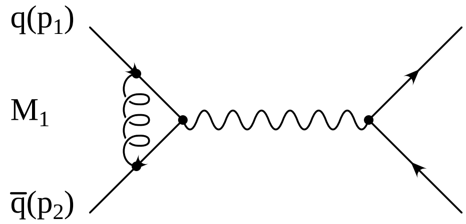
Therefore, the cross section diverges as

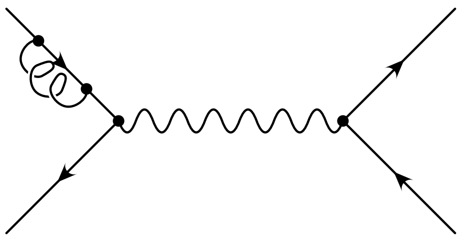
$$
\frac{\ln(s/p_T^2)}{p_T^2}
$$

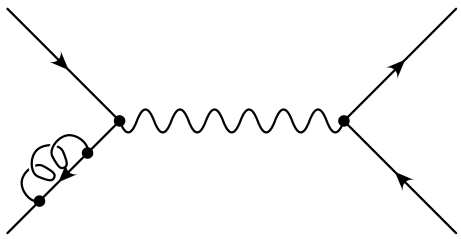
as $p_T^2 \to 0$.

Okay, so how do you calculate the $\mathcal{O}(\alpha_s)$ contribution to the cross section?









Both M_0 and M_1 have the same final state, so one calculates

$$|M_0 + M_1|^2 = |M_0|^2 + 2\text{Re } M_0^* M_1 + ...$$

Add these to the $q\bar{q} \to \gamma^* g$ and $qg \to \gamma^* q$ contributions.

- 1-loop graphs are divergent
- α_s tree graphs are divergent as $p_T \to 0$

Need a regularization scheme: choose dimensional regularization.

Divergences are converted to poles in $\epsilon = \frac{4-n}{2}$ dimensions.

Consider $\frac{d\sigma}{dQ^2}$. This is technically somewhat simpler than $\frac{d\sigma}{dQ^2 dy}$. Schematically, the $\mathcal{O}(\alpha_s)$ contributions take the following form:

$$
\begin{array}{ll}
\frac{d\sigma}{dQ^2} = \frac{d\sigma^0}{dQ^2} + \frac{\alpha_s}{2\pi}\left(\frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C\right) & q\bar{q} \to l^+ l^- \\
\frac{d\sigma}{dQ^2} = \frac{\alpha_s}{2\pi}\left(-\frac{A}{\epsilon^2} + \frac{B'}{\epsilon} + C'\right) & q\bar{q} \to l^+ l^- g \\
\frac{d\sigma}{dQ^2} = \frac{\alpha_s}{2\pi}\left(\frac{B''}{\epsilon} + C''\right) & qg \to l^+ l^- q
\end{array}
$$

- $\frac{1}{\epsilon^2}$ terms come from regions in phase space with both *soft* and *collinear* divergences.

- $\frac{1}{\epsilon}$ terms come from soft *or* collinear divergences.

- Soft divergences from the tree graphs cancel infrared divergences from the loop graphs.

- Remaining collinear divergences are related to the parton distributions.

$$
\frac{d\sigma}{dQ^2} = \frac{d\sigma^0}{dQ^2} + \frac{\alpha_s}{2\pi}\left(\frac{B + B' + B''}{\epsilon} + C + C' + C''\right)
$$

Detailed result:

$$
\begin{aligned}
\frac{d\sigma}{dQ^2} &= \frac{4\pi\alpha^2}{9Q^2}\int\frac{dx_a}{x_a}\int\frac{dx_b}{x_b}\left\{H_q(x_a,x_b,Q^2)\right.\\
&\quad\times\left[\delta(1-z)+\frac{\alpha_s(\mu^2)}{2\pi}\left(2P_{qq}(z)\left(-\frac{1}{\bar\epsilon}+\ln(\frac{Q^2}{\mu^2})\right)+D_q(z)\right)\right]\\
&+\frac{\alpha_s(\mu^2)}{2\pi}\left[G_{q/A}(x_a,Q^2)G_{g/B}(x_b,Q^2)+q\leftrightarrow g\right]\\
&\quad\left.\times\left[P_{qg}(z)\left(-\frac{1}{\bar\epsilon}+\ln(\frac{Q^2}{\mu^2})\right)+D_g(z)\right]\right\}
\end{aligned}
$$

where

$$\frac{1}{\bar\epsilon}=\frac{1}{\epsilon}+\ln 4\pi-\gamma_E \quad \text{and} \quad z=\tau/x_ax_b.$$

- The ϵ^{-2} soft-collinear poles have cancelled, but some ϵ^{-1} terms remain.

- These are the collinear poles associated with initial state radiation:

$$
\begin{array}{ll}
P_{qq}(z) & q\to qg\\
P_{qg}(z) & g\to q\bar q
\end{array}
$$

In the $\overline{\text{MS}}$ scheme we know how to define universal *scale dependent* parton distributions $G_i(x, \mu^2)$ in terms of the bare distributions $G_i(x)$:

$$G_q(x, \mu^2) = G_q(x) +$$
$$\frac{\alpha_s(\mu^2)}{2\pi}\left(-\frac{1}{\bar{\epsilon}}\right)\int_x^1 \frac{d\xi}{\xi}\left[P_{qq}\left(\frac{x}{\xi}\right)G_q(\xi) + P_{qg}\left(\frac{x}{\xi}\right)G_g(\xi)\right]$$

To see how this helps simplify the previous equation for $d\sigma/dQ^2$, we have to do a bit more work. Consider

$$\begin{aligned}
& \int \frac{dx_a}{x_a}\int\frac{dx_b}{x_b} G_q(x_a,\mu^2) G_{\bar{q}}(x_b,\mu^2)\delta(1-z) = \\
& \int \frac{dx_a}{x_a}\int\frac{dx_b}{x_b} G_q(x_a) G_{\bar{q}}(x_b)\delta(1-z) \\
- \quad & \frac{1}{\bar{\epsilon}}\frac{\alpha_s(\mu^2)}{2\pi}\int \frac{dx_a}{x_a}\int\frac{dx_b}{x_b} G_q(x_a) \\
& \times \int \frac{d\xi}{\xi}\left[P_{qq}\left(\frac{x_b}{\xi}\right)G_{\bar{q}}(\xi) + P_{qg}\left(\frac{x_b}{\xi}\right)G_g(\xi)\right]\delta(1-z) \\
+ \quad & (a \leftrightarrow b) + \mathcal{O}(\alpha_s^2)
\end{aligned}$$

where $z = \tau/x_a x_b$. Use the $\delta(1-\tau/x_a x_b)$ in the above equation to do the x_b integral.

Since $x_b = \tau/x_a$, the $\frac{1}{\bar{\epsilon}}$ line can be rewritten as

$$
\begin{aligned}
- & \frac{1}{\bar{\epsilon}} \frac{\alpha_s(\mu^2)}{2\pi} \int \frac{dx_a}{x_a} G_q(x_a) \\
& \times \int_{\tau/x_a}^{1} \frac{d\xi}{\xi} \left[P_{qq}\left(\frac{\tau}{x_a \xi}\right) G_{\overline{q}}(\xi) + P_{qg}\left(\frac{\tau}{x_a \xi}\right) G_g(\xi) \right]
\end{aligned}
$$

Relabel $\xi \to x_b$ to get

$$
\begin{aligned}
- & \frac{1}{\bar{\epsilon}} \frac{\alpha_s(\mu^2}{2\pi} \int \frac{dx_a}{x_a} \int \frac{dx_b}{x_b} G_q(x_a) G_{\overline{q}}(x_b) P_{qq}(z) \\
- & \frac{1}{\bar{\epsilon}} \frac{\alpha_s(\mu^2}{2\pi} \int \frac{dx_a}{x_a} \int \frac{dx_b}{x_b} G_q(x_a) G_g(x_b) P_{qg}(z)
\end{aligned}
$$

These are of the same form as the $1/\bar{\epsilon}$ terms in $d\sigma/dQ^2$. Replace

$$
G_q(x) \to G_q(x, \mu^2) + \frac{1}{\bar{\epsilon}} \ldots
$$

Then, the $1/\bar{\epsilon}$ terms cancel, leaving a finite expression for $d\sigma/dQ^2$. This is a demonstration of the factorization theorem at work. The collinear singularities associated with the pdf's are universal. Once the pdf's are defined using a factorization scheme, the cross sections are finite.

Using these results, the expression for $d\sigma/dQ^2$ can be simplified. In the $\overline{\text{MS}}$ scheme choosing $\mu^2 = Q^2$ we get:

$$\begin{aligned}
\frac{d\sigma}{dQ^2} &= \frac{4\pi\alpha^2}{9Q^2 s}\sum_q \int \frac{dx_a}{x_a}\int \frac{dx_b}{x_b} \\
&\times \left\{ H_q(x_a, x_b, Q^2)\left[\delta(1-z) + \frac{\alpha_s(Q^2)}{2\pi}D_q(z)\right]\right. \\
&+ \left[(G_q(x_a, Q^2) + G_{\overline{q}}(x_a, Q^2))G_g(x_b, Q^2) + (a \leftrightarrow b)\right] \\
&\times \left.\frac{\alpha_s(Q^2)}{2\pi}D_g(z)\right\}
\end{aligned}$$

with $z = \tau/x_a x_b$.

The $\delta(1-z)$ term reproduces the lowest order contribution. $D_q(z)$ and $D_g(z)$ give the finite $\mathcal{O}(\alpha_s)$ corrections. Note the presence of the *scale dependent* pdf's and the running coupling $\alpha_S(Q^2)$.

The earlier results on scaling will be modified due to the Q^2 dependence present in these results.

Factorization Schemes

The definition of the the scale dependent pdf's will affect the form of the D_q and D_g functions.

- $\overline{\text{MS}}$: the $1/\overline{\epsilon}$ terms given previously which come from the collinear singularities associated with the initial state radiation are combined with the bare pdf's to give the Q^2 dependent pdf's as has been shown above.

- DIS: additional finite terms are included along with the $1/\overline{\epsilon}$ parts so that the expression for $F_2(x,Q^2)$ in deep inelastic scattering retains its lowest order form when higher order terms are included.

In the $\overline{\text{MS}}$ scheme we have

$$\begin{aligned} D_q(z) &= C_F\left[4(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - 2\frac{1+z^2}{1-z}\ln z \right. \\ &\qquad \left. +\delta(1-z)\left(\frac{2\pi^2}{3}-8\right)\right] \\ D_g(z) &= T_R\left[(z^2+(1-z)^2)\ln\left(\frac{(1-z)^2}{z}\right)+\frac{1}{2}+3z-\frac{7}{2}z^2\right] \end{aligned}$$

Note: the gluon spin average is $\frac{1}{2(1-\epsilon)}$ in $n-$dimensions.

In the DIS scheme we get

$$\begin{aligned} D_q(z) &= C_F\left[2(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ + \frac{3}{(1-z)_+} - 6 - 4z \right. \\ &\qquad \left. +\delta(1-z)\left(1+\frac{4\pi^2}{3}\right)\right] \\ D_g(z) &= T_R\left[(z^2+(1-z)^2)\ln(1-z)+\frac{3}{2}-5z+\frac{9}{2}z^2\right] \end{aligned}$$

Comments on the $\mathcal{O}(\alpha_s)$ corrections

1. $\delta(1-z)\frac{\alpha_s}{2\pi}C_F\left(1+\frac{4\pi^2}{3}\right)$ in DIS

   - Part of the "π^2" term comes from the change from $Q^2 < 0$ for DIS to $Q^2 > 0$ for l^+l^- (spacelike virtual photon $\rightarrow$ timelike virtual photon)
   - phase space contains $(Q^2/\mu^2)^\epsilon \rightarrow (-Q^2/\mu^2)^\epsilon$
   - $(-1)^\epsilon = e^{\epsilon \ln(-1)} = 1 - \epsilon^2\pi^2/2 + \ldots$
   - Multiplied by $(-2/\epsilon^2 + \ldots)$ which gives a finite contribution proportional to π^2.

2. Phase space for DIS and l^+l^- production differ. $z \rightarrow 1$ corresponds to $\hat{s} = Q^2$ so the soft gluon singularities are at $z = 1$. There is a mismatch between the two phase spaces away from $z = 1$ so that the "+" regulator terms are different for the two cases.

These two items help to explain the size of the α_s corrections in l^+l^- relative to DIS.

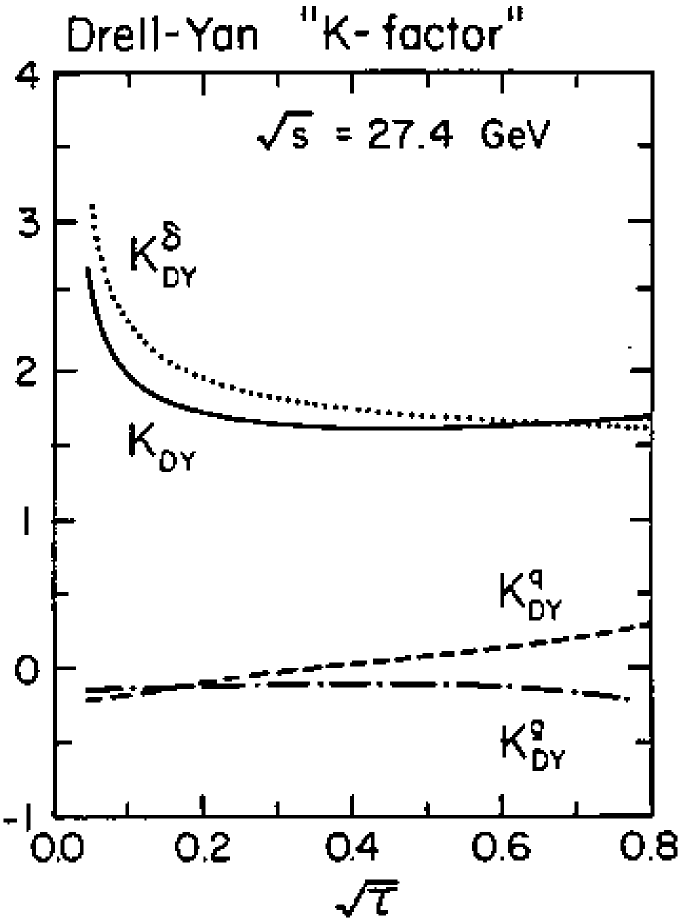


Figure 5.6 Drell-Yan "K-factor", K_{DY}, computed with $\Lambda = 200$ MeV at $\sqrt{s} = 27.4$ GeV plotted versus $\sqrt{\tau}$. The quark, gluon, and δ-function contributions are shown separately with $K_{DY} = K^{\delta}_{DY} + K^{q}_{DY} + K^{g}_{DY}$.

From "Applications of Perturbative QCD" by R.D. Field