Introduction

Monte Carlo event generators now an integral part of almost every experimental analysis.
All too often treated as a [redacted]

These lectures: concentrate on basic principles and physics issues.
Not a review of existing event generators.

Lecture 1:

1) Integrals as averages.
2) Multi-dimensional integration.
3) Phase space integration.
4) Event generators.
[5) NLO QCD programs.]

For anyone keen enough, there is a series of problems. Most need access to a computer, so cannot be done here. Solutions will appear at http://surya11.cern.ch/users/seymour/slides/ in due course. . .

Integrals as Averages

Basis of all Monte Carlo methods:

$$I = \int_{x_1}^{x_2} f(x)\, dx = (x_2 - x_1) \langle f(x) \rangle$$

Draw N values from a uniform distribution, x_i:

$$I \approx I_N \equiv (x_2 - x_1) \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

Sum invariant under reordering: randomize.

Central limit theorem: if x_i independent, I_N is a statistic with mean I and variance V/N,

$$V = (x_2 - x_1) \int_{x_1}^{x_2} \left[f(x)\right]^2 dx - \left[\int_{x_1}^{x_2} f(x)\, dx\right]^2$$

In practice: calculate Monte Carlo estimate of V, V_N, using the same sample x_i.

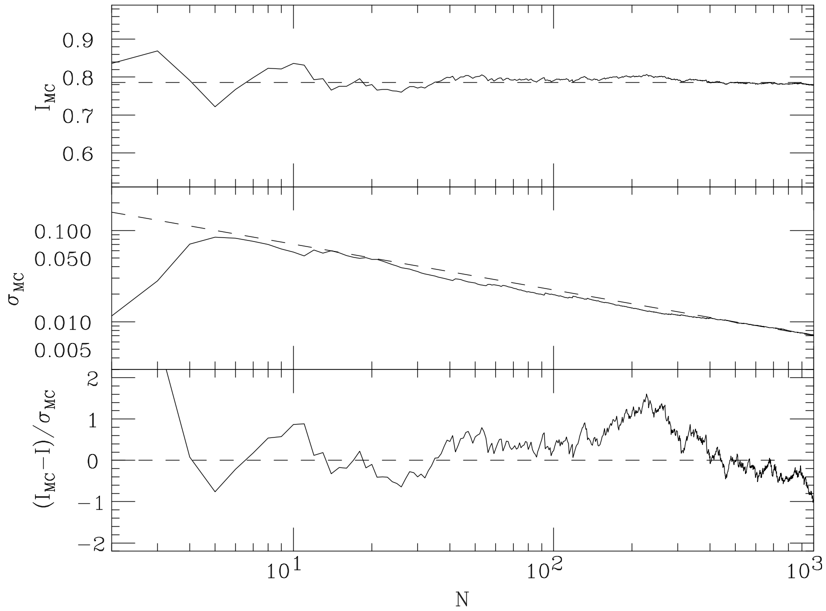
$$I \approx I_N \pm \sqrt{V_N/N}$$

Convergence

Monte Carlo integrals governed by the Central Limit Theorem $\Rightarrow$ error $\propto 1/\sqrt{N}$.

trapezium rule $\propto 1/N^2$,
Simpson's rule $\propto 1/N^4$,
etc

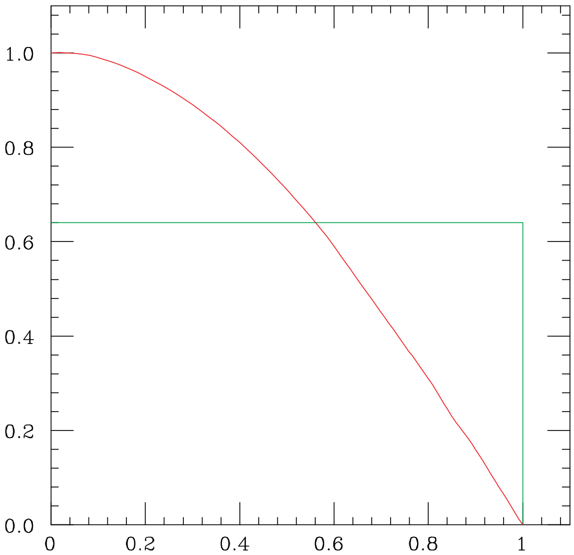
but only if derivatives exist and are finite
[c.f. $\sqrt{1-x^2} \sim 1/N^{3/2}$]:

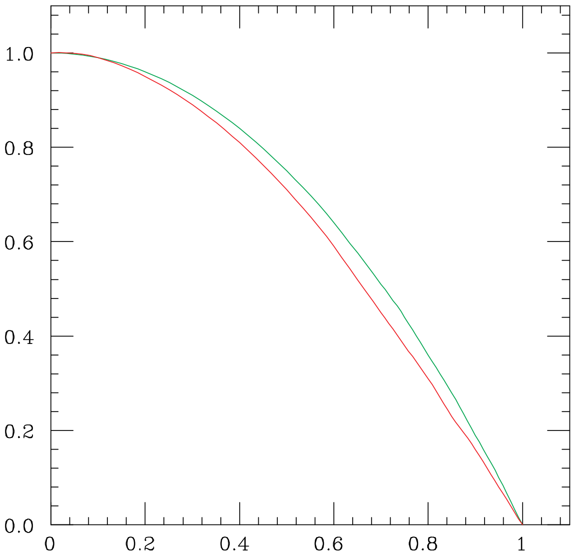


Write a Monte Carlo program to evaluate $\int_0^1 \sqrt{1-x^2}dx$ and its error. Compare it with the Central Limit values, as in this figure.

Importance Sampling

Convergence improved by putting more samples in region where function is largest. Corresponds to a Jacobian transformation.





$$\begin{aligned} I &= \int_0^1 dx \cos\tfrac{\pi}{2}x \\ &= 0.637 \pm 0.308/\sqrt{N} \end{aligned}$$

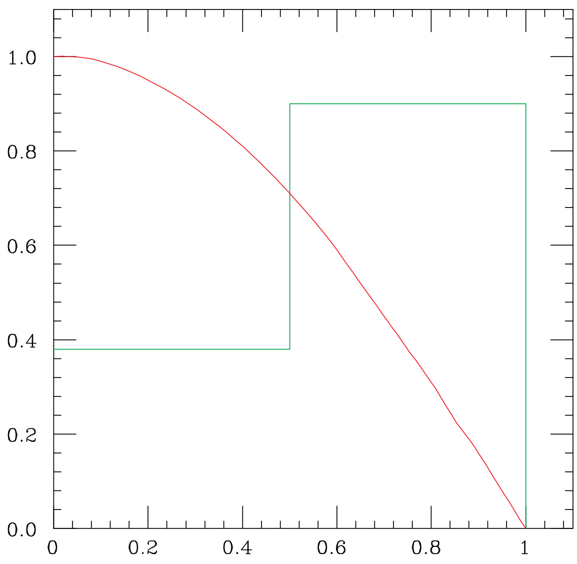
$$\begin{aligned} I &= \int_0^1 dx(1-x^2)\,\frac{\cos\frac{\pi}{2}x}{1-x^2} \\ &= \int d\rho\,\frac{\cos\frac{\pi}{2}x}{1-x^2}[x(\rho)] \\ &= 0.637 \pm 0.032/\sqrt{N} \end{aligned}$$

Derive the transformation from ρ to x, and check these results.

The Lazy Way

Divide up integration region piecemeal.
Can be done automatically (eg VEGAS).
Never as good as Jacobian transformations.

Optimize so that contribution to error from each region is equal $\Rightarrow$ total error minimized.



$$I = 0.637 \pm 0.147/\sqrt{N}$$

Not always intuitive!

Multi-dimensional Integration

Formalism extends trivially to many dimensions

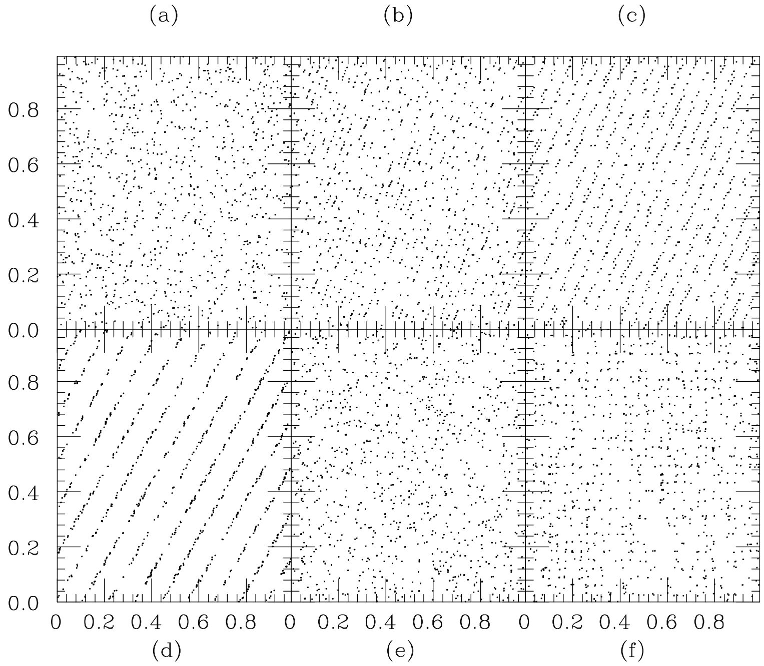
Particle physics: very many dimensions,
eg phase space = 3 dimensions per particles,
Tevatron event $\sim$ 200 hadrons.

Monte Carlo error remains $\propto 1/\sqrt{N}$.

trapezium rule $\propto 1/N^{2/d}$,
Simpson's rule $\propto 1/N^{4/d}$,
etc

Beware of correlations in 'random' numbers (the Marsaglia effect).
$\Rightarrow$ always use modern random number generators – F. James, Comp. Phys. Comm. 60 (1990) 329.

The Marsaglia Effect



Demonstration of the Marsaglia Effect in MLCGs. (a,b,c) are with the best multiplier for 16 bit machines ($a = 10125, m = 2^{16}, c = 1$): (a) is in two dimensions (no cut); (b) is in three dimensions with a cut of $Z = 1/10$; (c) is in four dimensions with a cut of $Z = 1/3$. (d) is with IBM's RANDU function ($a = 65539, m = 2^{29}, c = 0$) in 3 dimensions with a cut of $Z = 1/10$. (e,f) are the VAX library function MTH$RANDOM with one of the better multipliers for 32 bit machines ($a = 69069, m = 2^{32}, c = 1$): (e) is in five dimensions with a cut of $Z = 1/50$; (f) is in ten dimensions with a cut of $Z = 1/5$.

$$\rho_i = s_i/m$$

$$s_{i+1} = (a\, s_i + c)\mathrm{mod} m$$

(MLCG = Multiplicative Linear Congruential Generator = the way everyone used to do it!)

Summary

c.f. Gaussian quadrature.

Disadvantages of Monte Carlo:

- Slow convergence in few dimensions.

Advantages of Monte Carlo:

- Fast convergence in many dimensions.
- Arbitrarily complex integration regions (finite discontinuities not a problem).
- Few points needed to get first estimate ("feasibility limit").
- Every additional point improves accuracy ("growth rate").
- Easy error estimate.

Phase Space

$$\begin{aligned}\sigma &= \frac{1}{2s}\int |\mathcal{M}|^2 \, d\Pi_n(\sqrt{s}) \\ \Gamma &= \frac{1}{2M}\int |\mathcal{M}|^2 \, d\Pi_n(M)\end{aligned}$$

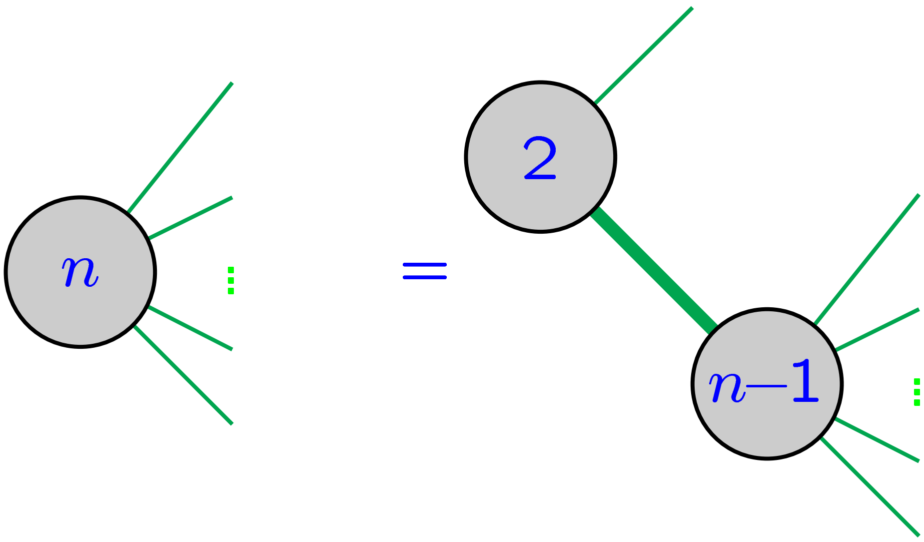
Phase space:

$$d\Pi_n(M) = \left[\prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3(2E_i)}\right](2\pi)^4\delta^{(4)}\left(p_0 - \sum_{i=1}^{n} p_i\right)$$

Two body phase space easy:

$$d\Pi_2(M) = \frac{1}{8\pi}\,\frac{2p}{M}\,\frac{d\Omega}{4\pi}$$

Other cases by recursive division:



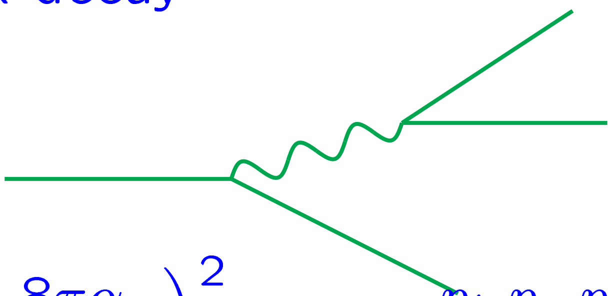
$$d\Pi_n(M) = \frac{1}{2\pi}\int_0^{(M-m)^2} dm_x^2 \, d\Pi_2(M) \, d\Pi_{n-1}(m_x)$$

Or by ‘democratic’ algorithms RAMBO, MAMBO
Can be better, but matrix elements rarely flat.

Particle Decays

Simplest application

eg top quark decay



$$|\mathcal{M}|^2 = \frac{1}{2}\left(\frac{8\pi\alpha}{\sin^2\theta_w}\right)^2 \frac{p_t\cdot p_\nu \; p_b\cdot p_\ell}{(m_W^2 - M_W^2)^2 + \Gamma_W^2 M_W^2}$$

$$\Gamma = \frac{1}{2M}\,\frac{1}{128\pi^3}\int |\mathcal{M}|^2 dm_W^2 \left(1-\frac{m_W^2}{M^2}\right)\frac{d\Omega}{4\pi}\,\frac{d\Omega_W}{4\pi}$$

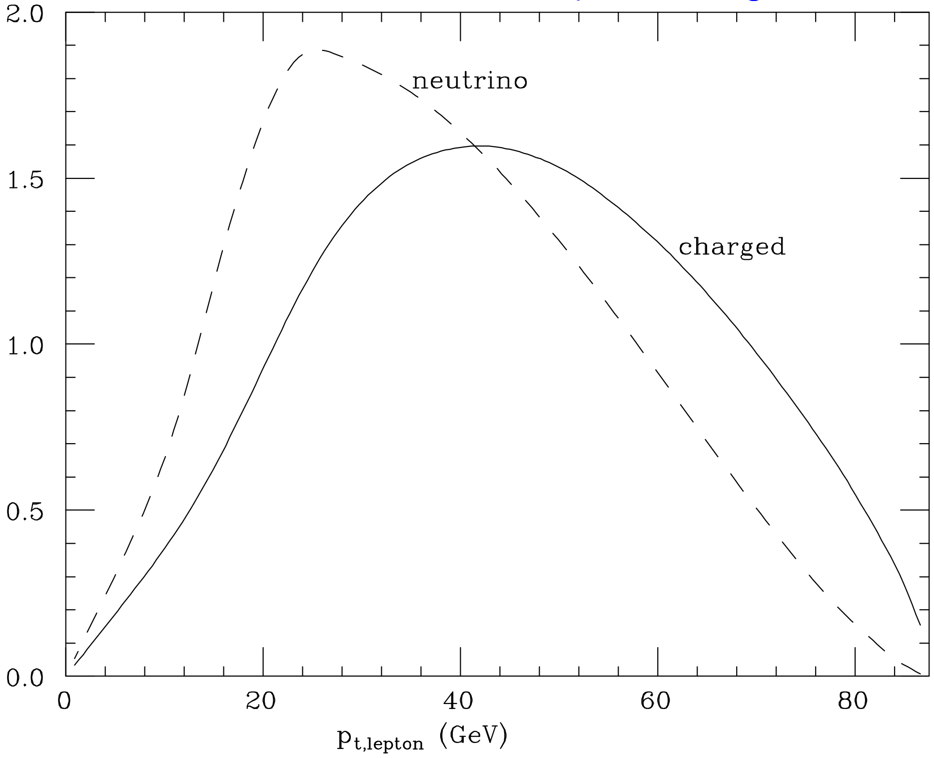
Breit-Wigner peak of W very strong
$\Rightarrow$ must be removed by Jacobian factor

Write a Monte Carlo program to implement this. Check how good the narrow-width approximation is.
(i.e. $1/[(m^2-M^2)^2+M^2\Gamma^2] \approx \pi/(M\Gamma)\delta(m^2-M^2)$)

Associated Distributions

Big advantage of Monte Carlo integration: simply histogram any associated quantities.

eg lepton momentum in top decays:



Almost any other technique requires new integration for each observable.

Can apply arbitrary cuts/smearing.

Cross Sections

Additional integrations over incoming parton densities:

$$\begin{aligned}\sigma(s) &= \int_0^1 dx_1 f_1(x_1) \int_0^1 dx_2 f_2(x_2)\, \hat{\sigma}(x_1 x_2 s) \\ &= \int_0^1 \frac{d\tau}{\tau} \hat{\sigma}(\tau s) \int_\tau^1 \frac{dx}{x}\, x f_1(x)\, \tfrac{\tau}{x} f_2(\tfrac{\tau}{x})\end{aligned}$$

$\hat{\sigma}(\hat{s})$ can have strong peaks, eg Z Breit-Wigner,
$\Rightarrow$ need Jacobian factors.
Hard to make process-independent.

eg top production, $q\bar{q} \to t\bar{t}$:

$$\begin{aligned}\frac{d\hat{\sigma}}{d\hat{t}} &= \frac{4}{9} \frac{\pi \alpha_s^2}{\hat{s}^4} \left[(m^2 - \hat{t})^2 + (m^2 - \hat{s} - \hat{t})^2 + 2m^2 \hat{s} \right] \\ \Rightarrow \hat{\sigma} &= \frac{4}{9} \frac{\pi \alpha_s^2}{\hat{s}} \frac{1}{3} \sqrt{1 - \frac{4m^2}{\hat{s}}} \left(2 + \frac{4m^2}{\hat{s}} \right).\end{aligned}$$

Strongly peaked close to $\hat{s} = 4m^2$.
Long tail $\sim 1/\hat{s}$.

Write a Monte Carlo for $q\bar{q} \to t\bar{t}$. Include your top decay routine. Calculate the full matrix elements for $q\bar{q} \to t\bar{t} \to b\bar{b}\ell\nu\ell'\nu'$. How good an approximation is it to neglect the top decay correlations?

Event Generators

Up to here, only considered Monte Carlo as a numerical integration method.
If function being integrated is a probability density (positive definite), trivial to convert it to a simulation of physical process
= an event generator.

Simple example:

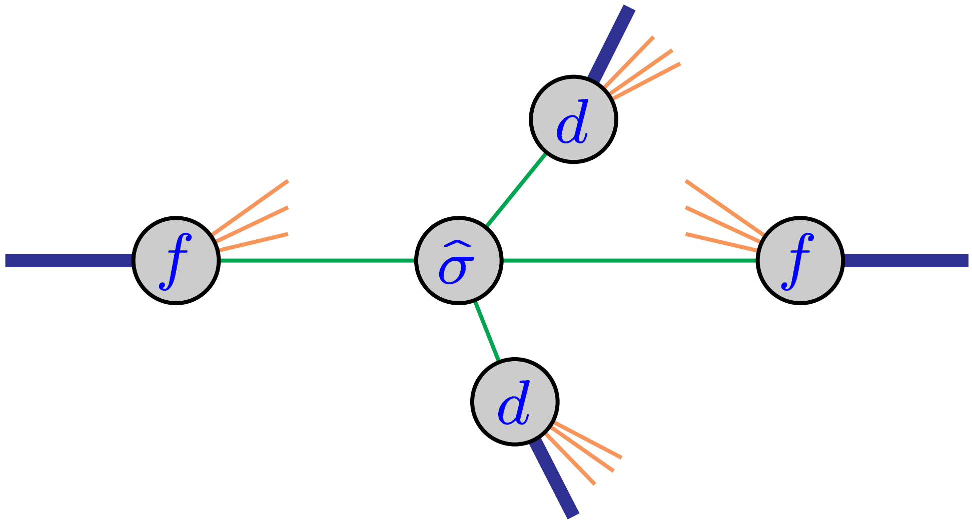
$$\sigma = \int_0^1 \frac{d\sigma}{dx}\,dx$$

Naive approach: 'events' x, 'weights' $d\sigma/dx$.
Can generate unweighted events by keeping them with probability $(d\sigma/dx)/(d\sigma/dx)_{\rm max}$, and give them all weight $\sigma_{\rm tot}$.
$\Rightarrow$ happen with same frequency as in nature.

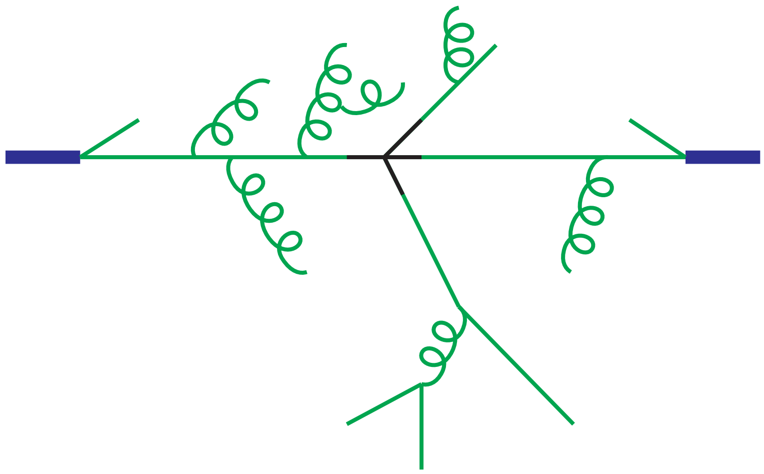
Efficiency: $\frac{(d\sigma/dx)_{\rm avge}}{(d\sigma/dx)_{\rm max}}$ = fraction of generated events kept.

QCD event generators

Theorist's eye view of jet production:



Compared to QCD event generator's:



1) Hard process
2) Parton shower
3) Hadronization

NLO QCD calculations

Two separate divergent integrals:

$$\sigma_{NLO} = \int_{m+1} d\sigma^R + \int_m d\sigma^V.$$

Must combine before numerical integration.

Jet definition could be arbitrarily complicated.

$$d\sigma^R = d\Pi_{m+1} \, |\mathcal{M}_{m+1}|^2 \, F^J_{m+1}(p_1, \ldots, p_{m+1}).$$

How to combine without knowing F^J?

Two solutions:
phase space slicing and subtraction method.

Illustrate with simple one-dim. example:

$$|\mathcal{M}_{m+1}|^2 \equiv \frac{1}{x}\mathcal{M}(x)$$

x = gluon energy or two-parton invariant mass.
Divergences regularized by $d = 4 - 2\epsilon$ dims:

$$|\mathcal{M}_m^{\text{one-loop}}|^2 \equiv \frac{1}{\epsilon}\mathcal{V}$$

Cross section in d dimensions is:

$$\sigma = \int_0^1 \frac{dx}{x^{1+\epsilon}}\,\mathcal{M}(x)\;F_1^J(x) + \frac{1}{\epsilon}\mathcal{V}\;F_0^J.$$

Infrared safety: $F_1^J(0) = F_0^J$.
KLN cancellation theorem: $\mathcal{M}(0) = \mathcal{V}$.

Phase space slicing:

arbitrary cutoff $\delta \ll 1$:

$$\begin{aligned}
\sigma &= \int_0^\delta \frac{dx}{x^{1+\epsilon}}\,\mathcal{M}(x)\;F_1^J(x) + \int_\delta^1 \frac{dx}{x^{1+\epsilon}}\,\mathcal{M}(x)\;F_1^J(x) + \frac{1}{\epsilon}\mathcal{V}\;F \\
&\approx \int_0^\delta \frac{dx}{x^{1+\epsilon}}\,\mathcal{V}\;F_0^J + \int_\delta^1 \frac{dx}{x}\,\mathcal{M}(x)\;F_1^J(x) + \frac{1}{\epsilon}\mathcal{V}\;F_0^J \\
&= \int_\delta^1 \frac{dx}{x}\,\mathcal{M}(x)\;F_1^J(x) + \log(\delta)\mathcal{V}\;F_0^J
\end{aligned}$$

two separate finite integrals $\Rightarrow$ Monte Carlo.

Becomes exact for $\delta \to 0$, but numerical errors blow up.

$\Rightarrow$ compromise (= trial and error).

Systematized by Giele-Glover-Kosower.

JETRAD, DYRAD, EERAD, . . . , MEPJET.

Subtraction method:

exact identity:

$$
\begin{aligned}
\sigma &= \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{M}(x)\, F_1^J(x) - \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{V}\, F_0^J \\
&\qquad + \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{V}\, F_0^J + \frac{1}{\epsilon}\mathcal{V}\, F_0^J \\
&= \int_0^1 \frac{dx}{x} \left(\mathcal{M}(x)\, F_1^J(x) - \mathcal{V}\, F_0^J\right) + \mathcal{O}(1)\, \mathcal{V}\, F_0^J.
\end{aligned}
$$

Two separate finite integrals again.

Much harder – subtracted cross section must be valid everywhere in phase space.

eg Ellis-Kunszt-Soper jet program.

Systematized in

S. Catani and M.H. Seymour, Nucl. Phys. B485 (1997) 291.

– any observable in any process

– analytical integrals done once-and-for-all

EVENT2, DISENT.

Use the gluon-mass regularization scheme to construct a NLO Monte Carlo program for the two-jet rate in e^+e^- annihilation.
The real cross section is:

$$
\begin{aligned}
d\sigma^R &= C_F \frac{\alpha_s}{2\pi} \int_0^1 dx_1 dx_2 \Theta((1-x_1)(1-x_2)-\epsilon)\Theta(x_1+x_2+\epsilon-1) \\
&\quad \left\{ \frac{(x_1+\epsilon)^2+(x_2+\epsilon)^2}{(1-x_1)(1-x_2)} - \frac{\epsilon}{(1-x_1)^2} - \frac{\epsilon}{(1-x_2)^2} \right\},
\end{aligned}
$$

where $\epsilon = m_g^2/s$.
The virtual cross section is:

$$
\sigma^V = C_F \frac{\alpha_s}{2\pi} \left\{ -\log^2\epsilon - 3\log\epsilon - \frac{7}{2} + \frac{\pi^2}{3} + \mathcal{O}(\epsilon) \right\}.
$$

Calculate the inclusive jet rate as a function of their direction in the lab using several different jet definitions.

Summary

Monte Carlo is a very convenient numerical integration method.
Well-suited to particle physics: difficult integrands, many dimensions.

Integrand positive definite $\rightarrow$ event generator.

Fully exclusive $\rightarrow$ treat particles exactly like in data.

$\Rightarrow$ need to understand/model hadronic final state.

[NLO QCD programs are not event generators.
Not positive definite.
But full numerical treatment of arbitrary observables.]